ISOTOPIC GENERALIZATION OF THE LEGENDRE, JACOBI, AND BESSEL FUNCTIONS

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Received June 25, 1995
Algebras, Groups & Geometries **12** (1995) 255-305.

Abstract

In this paper, we consider the Lie-isotopic generalizations of the Legendre, Jacobi, and Bessel functions.

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Contents

1 Introduction

In this paper, we consider the Lie-isotopic generalization of the Legendre, Jacobi, and Bessel functions.

We describe in detail the group of rotations of three-dimensional isoEuclidean space, and the group locally isomorphic to it, $\hat{SU}(2)$, consisting of isounitary isounimodular 2×2 matrices. Also, we study the group $\hat{QU}(2)$ of quasiunitary matrices and the group $\hat{M}(2)$ of isometric transformations of isoEuclidean plane.

These studies are of interest both in mathematical and physical points of view. We refer the interested reader to monographs [3] for comprehensive review on the Lie-isotopic formalism and its applications.

The isotopic generalizations of the groups $SO(3)$, $SU(2)$, and $M(2)$ are of continuing interest in the literature. From physical point of view, our interest is that the Lie-isotopic generalizations of the Legendre functions as well as the other special functions, such as Jacobi and Bessel functions, can be used in formulating the nonpotential scattering theory [1, 2, 6, 7] when one considers non-zero isoangular momenta.

The paper is organized as follows.

Sections 2-7 are devoted to representations of the group $\hat{SU}(2)$ and isoLegendre functions. Namely, in Sec.2, we consider the group $\hat{SU}(2)$. In Sec.3, we consider unitary irreducible representations (irreps) of the group $\hat{SU}(2)$. In Sec.4, we present matrix elements of the unitary irreps of $\hat{SU}(2)$, and isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$. In Sec.5, we present basic properties of the isoLegendre functions. In Sec.6, we present functional relations satisfied by the isoLegendre functions. In Sec.7, we present recurrency relations satisfied by the isoLegendre functions.

Sections 8-14 are devoted to representations of the group $\hat{QU}(2)$ and isoJacobi functions.

Sections 15-20 are devoted to representations of the group $\hat{M}(2)$ and isoBessel functions.

2 The group $\hat{SU}(2)$

In this Section, we consider representations of the group $\hat{SU}(2)$, elements of which are isounitary isounimodular 2×2 matrices, and its relation to the group $\hat{SO}(3)$ of rotations of three dimensional isoEuclidean space.

2.1 Parametrizations

Denote $\hat{SU}(2)$ the set of isounitary isounimodular 2×2 matrices, namely, of the matrices

$$\hat{u} = \hat{I} * \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\gamma}} & \hat{\delta} \end{pmatrix}. \tag{1}$$

If $\hat{u}_1 \in \hat{SU}(2)$ and $\hat{u}_2 \in \hat{SU}(2)$ then

$$(\hat{u}_1 * \hat{u}_2)^* = \hat{u}_2^* * \hat{u}_1^* = \hat{u}_2^{-1} * \hat{u}_1^{-1} = (\hat{u}_1 * \hat{u}_2)^{-1} \tag{2}$$

and $\det(\hat{u}_1 * \hat{u}_2) = 1$. Therefore, $\hat{u}_1 * \hat{u}_2 \in \hat{SU}(2)$. Also, it is easy to show that $\hat{u}_1^{-1} \in \hat{SU}(2)$. We arrive at the conclusion that $\hat{SU}(2)$ is a *group*.

Let $\hat{u} \in \hat{SU}(2)$. Since

$$\hat{u}^* = \hat{I} * \begin{pmatrix} \bar{\hat{\alpha}} & \bar{\hat{\beta}} \\ \bar{\hat{\gamma}} & \bar{\hat{\delta}} \end{pmatrix} \tag{3}$$

and

$$\hat{u}^{-1} = \hat{I} * \begin{pmatrix} \hat{\delta} & -\hat{\beta} \\ -\bar{\hat{\gamma}} & \hat{\alpha} \end{pmatrix}, \tag{4}$$

then $\hat{\delta} = \bar{\hat{\alpha}}$ and $\hat{\gamma} = -\bar{\hat{\beta}}$.

Thus, any matrix $\hat{u} \in \hat{SU}(2)$ has the form

$$\hat{u} = \hat{I} * \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ -\bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}, \tag{5}$$

where $\hat{I} = \mathrm{diag}(g_{11}^{-1}, g_{22}^{-1})$, $\det I = \Delta$. Since $\det \hat{u} = 1$ then

$$|\hat{\alpha}|\Delta|\hat{\alpha}| + |\hat{\beta}|\Delta|\hat{\beta}| = 1, \tag{6}$$

and vice versa, if $\hat{u}$ is a matrix of the form (5) and Eq.(6) holds, then $\hat{u} \in \hat{SU}(2)$.

From the above consideration, it follows that the elements of $\hat{SU}(2)$ can be uniquely determined by two complex numbers $(\hat\alpha, \hat\beta)$ obeying eq. (6). These complex numbers can be presented by three real parameters, for example, by $|\hat\alpha|$, $\mathrm{Arg}\hat\beta$, and $\mathrm{Arg}\hat\alpha$. If $\hat\alpha * \hat\beta \neq 0$, one can use another parametrization, namely Euler angles, $\hat\varphi$, $\hat\theta$, and $\hat\psi$, which are related to $|\hat\alpha|$, $\arg\hat\beta$, and $\arg\hat\alpha$ according to the following relations:

$$|\hat\alpha| = g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2] \equiv \mathrm{isocos}[\hat\theta/2],$$

$$\mathrm{Arg}\hat\alpha = \frac{\hat\varphi + \hat\psi}{2}, \quad \mathrm{Arg}\hat\beta = \frac{\hat\varphi - \hat\psi + \pi}{2}, \tag{7}$$

where

$$\hat\varphi \equiv \varphi\Delta^{1/2}, \quad \hat\theta \equiv \theta\Delta^{1/2}, \quad \hat\psi \equiv \psi\Delta^{1/2}. \tag{8}$$

The values of the Euler angles are not determined by (7) uniquely, so that we must put additionally

$$0 \le \hat\varphi < 2\pi, \quad 0 < \hat\theta < \pi, \quad -2\pi \le \hat\psi < 2\pi. \tag{9}$$

From (7) it follows that $|\hat\beta| = g_{22}^{-1/2}\sin(\theta\Delta^{1/2}/2)$ and that the matrix $\hat u = \hat u(\hat\varphi, \hat\theta, \hat\psi)$ has the following form:

$$\hat u = \begin{pmatrix} gg_{11}^{-1/2}cos\hat\theta/2\Delta e^{i(\hat\varphi+\hat\psi)/2} & i\Delta^2 g_{11}^{-1/2}\sin\hat\theta/2e^{i(\hat\varphi-\hat\psi)/2} \\ i\Delta^2 g_{22}^{-1/2}\sin\hat\theta/2e^{-i(\hat\psi-\hat\varphi)/2} & gg_{11}^{-1/2}\cos\hat\theta/2\Delta e^{-i(\hat\varphi+\hat\psi)/2} \end{pmatrix}. \tag{10}$$

From (5) and (10) we have

$$g_1 1^{-1/2}\cos[\psi\Delta^{1/2}] = 2|\alpha|^2\Delta - 1,$$

$$\exp[i\Delta^{1/2}\psi/2] = -i\frac{\alpha\beta}{|\alpha|\beta|}, \tag{11}$$

$$\exp i\Delta^{1/2}\psi/2 = \frac{\alpha\Delta^{1/2}\exp\{-i\Delta^{1/2}\varphi/2\}}{|\alpha|}.$$

Also, from (10) we have the following factorization

$$\hat u(\hat\varphi, \hat\theta, \hat\psi) = \begin{pmatrix} \exp\{i\Delta^{1/2}\varphi/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}\varphi/2\} \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} g_{11}^{-1/2}\cos\hat\theta/2 & i\Delta g g_{22}^{-1/2}sin\hat\theta/2 \\ g_{22}^{-1/2}\sin\hat\theta/2 & i\Delta g_{11}^{-1/2}\cos\hat\theta/2 \end{pmatrix} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

$$\times\begin{pmatrix}\exp\{i\Delta^{1/2}\psi/2\} & 0\\ 0 & \exp\{-i\Delta^{1/2}\psi/2\}\end{pmatrix}$$

$$\equiv \hat{u}(\hat{\varphi},0,0)\Delta\hat{u}(0,\hat{\theta},0)\Delta\hat{u}(0,0,\hat{\psi}). \tag{12}$$

Diagonal matrices

$$\begin{pmatrix}\exp[i\Delta^{1/2}\varphi/2] & 0\\ 0 & \exp[-i\Delta^{1/2}\varphi/2]\end{pmatrix} \tag{13}$$

form a one-parameter subgroup of $\hat{SU}(2)$. Thus every matrix $\hat{u}\in\hat{SU}(2)$ lies in the left and right conjugacy class in respect to ths subgroup containing the matrix of the form

$$\begin{pmatrix} g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2}\sin[\theta\Delta^{1/2}/2]\\ i\Delta g_{22}^{-1/2}\sin[\theta\Delta^{1/2}/2] & g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2]\end{pmatrix}. \tag{14}$$

Note that the matrices represented by (14) form a one-parameter subgroup of $\hat{SU}(2)$.

2.2 IsoEuler angles for matrix product

Let $\hat{u} = \hat{u}_1\Delta\hat{u}_2$ is a product of two matrices $\hat{u}_1, \hat{u}_2 \in \hat{SU}(2)$. Denote the corresponding isoEuler angles by $(\hat{\varphi},\ \hat{\theta},\ \hat{\psi})$, $(\hat{\varphi}_1,\ \hat{\theta}_1,\ \hat{\psi}_1)$, and $(\hat{\varphi}_2,\ \hat{\theta}_2,\ \hat{\psi}_2)$. To express the isoEuler angles of $\hat{u}$ via the isoEuler angles of $\hat{u}_1$ and $\hat{u}_2$, we consider the case when $\hat{\varphi}_1 = \hat{\psi}_1 = \hat{\psi}_2 = 0$. For this case we have

$$\hat{u} = \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\theta}_1/2 & i\Delta g_{22}^{-1/2}\sin\hat{\theta}_1/2\\ i\Delta g_{22}^{-1/2}\sin\hat{\theta}_1/2 & g_{11}^{-1/2}\cos\hat{\theta}_1/2\end{pmatrix}\begin{pmatrix} g_{11} & 0\\ 0 & g_{22}\end{pmatrix}$$

$$\times\begin{pmatrix} g_{11}^{-1/2}\Delta\cos\hat{\theta}_2/2\exp\{i\hat{\varphi}_2/2\} & i\Delta^2\sin\hat{\theta}_2/2\exp\{i\hat{\varphi}_2/2\}\\ i\Delta^2\sin\hat{\theta}_2/2\exp\{-i\hat{\varphi}_2/2\} & g_{11}^{-1/2}\Delta\cos\hat{\theta}_2/2\exp\{-i\hat{\varphi}_2/2\}\end{pmatrix}. \tag{15}$$

Using (11) we have from (15) in sequence

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]g_{11}^{-1/2}$$

$$-\sin[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta g g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}], \tag{16}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{(g_1 1)^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{i\sin[\theta_2\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{17}$$

$$\exp\{i\Delta^{1/2}(\varphi+\psi)/2\} = \frac{g_{11}^{-1}\Delta\cos[\theta_2\Delta^{1/2}/2]\Delta^2\exp\{i\Delta^{1/2}\varphi_2/2\}}{\cos[\theta_1\Delta^{1/2}/2]}$$

$$-\frac{g_{22}^{-1/2}\sin[\theta_1\Delta^{1/2}/2]\Delta\sin[\theta_2\Delta^{1/2}/2]\Delta\exp\{-i\Delta^{1/2}\varphi_2/2\}}{g_{11}^{-1/2}\cos[\theta\Delta^{1/2}/2]}. \tag{18}$$

It is more convenient to use the following expressions:

$$\Big(\frac{g_{22}}{g_{11}}\Big)^{-1/2}\tan[\varphi\Delta^{1/2}]$$

$$= \frac{\sin[\theta_2\Delta^{1/2}]\Delta\sin[\psi_2\Delta^{1/2}]}{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta\cos[\varphi_2\Delta^{1/2}]+g_{11}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]}, \tag{19}$$

$$\Big(\frac{g_{22}}{g_{11}}\Big)^{-1/2}\tan[\psi\Delta^{1/2}]$$

$$= \frac{\sin[\theta_1\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}]\Delta\cos[\psi_2\Delta^{1/2}]+g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]}. \tag{20}$$

Due to the results of this particular case can easily turn to the general case. Indeed, according to (12) we have

$$\hat{u}(\hat{\varphi}_1,\hat{\theta}_1,\hat{\psi}_1)\Delta\hat{u}(\hat{\varphi}_2,\hat{\theta}_2,\hat{\psi}_2))$$

$$= \Delta^5\hat{u}(\hat{\varphi}_1,0,0)\hat{u}(0,\hat{\psi}_1,0)\hat{u}(0,0,\hat{\psi}_1)\hat{u}(\hat{\varphi}_2,0,0)\hat{u}(0,\hat{\theta}_2,0)\hat{u}(0,0,\hat{\varphi}_2). \tag{21}$$

Note that

$$\hat{u}(0,0,\hat{\psi}_1)\Delta\hat{u}(\hat{\varphi}_2,0,0) = \hat{u}(\hat{\varphi}_2+\hat{\psi}_1,0,0). \tag{22}$$

We observe that the result of the product $\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi}) * \hat{u}(\hat{\varphi}_1,0,0)$ gives the matrix $\hat{u}(\hat{\varphi}+\hat{\varphi}_1,\hat{\theta},\hat{\psi})$. Similarly, the result of the product $\hat{u}(0,0,\hat{\psi}_1) * \hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi})$ gives the matrix $\hat{u}(\hat{\varphi},\hat{\theta},\hat{\psi}+\hat{\psi}_1$. From these observations it follows that the formulas (16)-(18) are valid in general case with the replacements $\hat{\varphi}_2 \to \hat{\varphi}_2+\hat{\psi}_1$, $\hat{\varphi} \to \hat{\varphi}-\hat{\varphi}_1$, and $\hat{\psi} \to \hat{\psi}-\hat{\psi}_2$.

Namely, in an explicit form

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$

$$-gg_{22}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\cos[(\varphi_2+\psi_1)\Delta^{1/2}], \quad (23)$$

$$\exp\{iD\varphi\} = \frac{g_{11}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta\cos[(\psi_2+\psi_1)\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{ig_{22}^{-1/2}\sin[\theta_2\Delta^{1/2}]\Delta\sin[(\varphi_2+\psi_1)\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \quad (24)$$

$$\exp\{i\Delta^{1/2}(\varphi-\varphi_1+\psi-\psi_2)/2\}$$

$$= \frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}/2]\Delta\cos[\theta_2\Delta^{1/2}/2]\Delta\exp\{i\Delta^{1/2}(\varphi_2+\psi_1)/2\}}{g_{11}^{-1/2}\cos[\psi\Delta^{1/2}/2]}$$

$$-\frac{g_{22}^{-1/2}\sin[\psi_1\Delta^{1/2}/2]\Delta\sin[\theta_2\Delta^{1/2}/2]\Delta\exp\{-i\Delta^{1/2}(\varphi_2+\psi_1)/2\}}{g_{11}^{-1/2}\cos[\psi\Delta^{1/2}/2]}. \quad (25)$$

2.3 Relation to the group of rotations

Let us define the relation between the groups $\hat{SU}(2)$ and $\hat{SO}(3)$. To this end, we identify the vector $\hat{x}(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ of three dimensional isoEuclidean space with the complex 2 matrix of the form

$$\hat{h}_x = \begin{pmatrix} \hat{x}_3 & \hat{x}_1+i\hat{x}_2 \\ \hat{x}_1-i\hat{x}_2 & -\hat{x}_3 \end{pmatrix}, \quad (26)$$

where $\hat{x} = x\hat{I} = x\delta^{-1}$. The set of the matrices of the form (26) consists of isoHermitean matrices $\hat{g}$ with $\mathrm{Tr}\hat{g}=0$. Namely, with every matrix $\hat{u}\in \hat{SU}(2)$ we relate the transformation $\hat{T}(\hat{u})$,

$$\hat{T}(\hat{u})\Delta\hat{h}_x = \hat{u}\Delta\hat{h}_x * \hat{u}. \quad (27)$$

Since for the isounitary matrices we have $\hat{u}^* = \hat{u}^{-1}$, the traces of $\hat{h}_x$ and $\hat{T}(\hat{u})\Delta\hat{h}_x$ coincide so that the trace of $\hat{T}(\hat{u})\Delta\hat{h}_x$ is zero. Also, we have

$$(\hat{T}(\hat{u})\Delta\hat{h}(\hat{x}))^* = (\hat{u}\Delta\hat{h}_x\Delta*\hat{u})^* = \hat{u}\Delta*\hat{h}_x*\hat{u}\Delta = \hat{u}\Delta\hat{h}_x*\hat{u} = \hat{T}(\hat{u})\Delta\hat{h}_x, \quad (28)$$

so that the matrix $\hat{T}(\hat{u})\Delta\hat{h}_x$ is indeed isoHermitean. On the other hand, for isoHermitean matrices we have the following representation:

$$\hat{T}(\hat{u})\Delta\hat{h}_x = \begin{pmatrix} \hat{y}_3 & \hat{y}_1+i\Delta\hat{y}_2 \\ \hat{y}_1-i\Delta\hat{y}_2 & -\hat{y}_3 \end{pmatrix} \equiv \begin{pmatrix} \Delta^{-1}y_3 & \Delta y_1+iy_2 \\ \Delta^{-1}y_1-iy_2 & -\Delta y_3 \end{pmatrix} = \hat{h}_y, \quad (29)$$

where $\hat{y}(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ is a vector in three dimensional isoEuclidean space.

From (27) it can be seen that the components of $\hat{y}$ are linear combinations of the components of $\hat{x}$ so that $\hat{T}(\hat{u})$ is a linear transformation of the three dimensional isoEuclidean space $\hat{E}^3$. From the local isomorphism between the groups $\hat{SU}(2)$ and $\hat{SO}(3)$ it follows that rotations of $\hat{E}^3$ can be parametrized by the isoEuler angles $(\hat{\varphi},\, \hat{\theta},\, \hat{\psi})$. Here, the angle $\hat{\varphi}$ varies from 0 to 2π since $\hat{u}$ and $-\hat{u}$ correspond to the same rotation.

Due to (12) the matrices $\hat{u}(\hat{\varphi}, 0, 0)$ and $\hat{u}(0, 0, \hat{\psi})$ can be presented as

$$\begin{pmatrix} \exp\{i\Delta^{1/2}t/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}t/2\} \end{pmatrix} \equiv \hat{\omega}_3(\hat{t}), \tag{30}$$

where $\hat{\omega}_3(\hat{t})$ is the rotation by the angle $\hat{t}$ around the axis $O\hat{x}_3$, and $\hat{u}(0, \hat{\theta}, 0)$ has the form

$$\begin{pmatrix} g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2}\sin[t\Delta^{1/2}/2] \\ i\Delta(g_{22})^{-1/2}\sin[t\Delta^{1/2}/2] & g_{11}^{-1/2}\cos[t\Delta^{1/2}/2] \end{pmatrix} \equiv \hat{\omega}_1(\hat{t}), \tag{31}$$

which is the rotation around the axis $O\hat{x}_1$. From this observation, we have the following decomposition for arbitrary rotation $\hat{g}$ of $\hat{E}^3$:

$$\hat{g}(\hat{\varphi}, \hat{\theta}, \hat{\psi}) = \hat{g}(\hat{\varphi}, 0, 0)\Delta\hat{g}(0, \hat{\theta}, 0)\Delta\hat{g}(\hat{\psi}, 0, 0)$$

$$= \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\varphi} & -g_{22}^{-1/2}\sin\hat{\varphi} & 0 \\ g_{22}^{-1/2}\sin\hat{\varphi} & g_{11}^{-1/2}\cos\hat{\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

$$\times \begin{pmatrix} \Delta^{-1} & 0 & 0 \\ 0 & g_{11}^{-1/2}\cos\hat{\theta} & -g_{22}^{-1/2}\sin\hat{\theta} \\ 0 & (g_{22})^{-1/2}\sin\hat{\psi} & (g_{11})^{-1/2}\cos\hat{\theta} \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

$$\times \begin{pmatrix} \Delta^{-1} & 0 & 0 \\ 0 & g_{11}^{-1/2}\cos\hat{\psi} & -g_{22}^{-1/2}\sin\hat{\psi} \\ 0 & (g_{22})^{-1/2}\sin\hat{\psi} & (g_{11})^{-1/2}\cos\hat{\psi} \end{pmatrix}. \tag{32}$$

3 Irreps of $\hat{SU}(2)$

Recall that with any isounimodular complex 2×2 matrix $\hat{g}$ we associate the linear transformation,

$$\hat{w}_1 = \alpha\Delta^2 z_1 + \gamma\Delta^2 z_2 = \Delta^2(\alpha z_1 + \gamma z_2), \tag{33}$$

$$\hat{w}_2 = \beta\Delta z_1 + \delta\delta^2 z_2 = \Delta^2(\beta z_1 + \delta z_2),$$

of two dimensional linear complex space. Such a transformation can be presented by the operator,

$$\hat{T}(\hat{g})\Delta\hat{f}(\hat{z}_1, \hat{z}_2) = \hat{f}(\hat{\alpha}\Delta\hat{z}_1 + \hat{\gamma}\Delta\hat{z}_2; \hat{\beta}\Delta\hat{z}_1 + \hat{\Delta}\hat{z}_2), \qquad (34)$$

acting on the space of functions of two complex variables. Evidently,

$$\hat{T}(\hat{g}_1\Delta\hat{g}_2) = \hat{T}(\hat{g}_1) + \hat{T}(\hat{g}_2),$$

so that $\hat{T}(\hat{g})$ is a *representation* of the group $\hat{SL}(2, C)$. Similarly to the theorem from Ref.[4] we have the following

Proposition 1. Every irreducible isounitary representation $\hat{T}(\hat{u})$ of $\hat{SU}(2)$ is equivalent to one of the representations $\hat{T}_l(\hat{u})$, where $l = 0, 1/2, 1, \ldots$.

The prove of the Proposition 1 is analogous to that of given in Ref.[4], and we do not present it here.

From Proposition 1 it follows that in the space of subgroup $\hat{SU}(2)$ there exists the orthogonal normalized basis, $\hat{f}_{-l}, \cdots, \hat{f}_l$, such that the operators $\hat{T}(\hat{u})$ are represented in this basis by the same matrices as the operators $\hat{T}_l(\hat{u})$ in the basis $\{\hat{\psi}_k(x)\}$, where

$$\hat{\psi}_k(x) = \Delta^{-s+1/2}\frac{x^{l-k}}{\sqrt{(l-k)!(l+k)!}}, \qquad (35)$$

$$-l \le k \le l, \quad s = 1, \ldots, n.$$

We call such a basis *isocanonical*. It is easy to verify that isocanonical basis is determined uniquely up to scalar factor λ, with $|\lambda| = \Delta^{-1}$. More precisely, isocanonical basis consists of normalized eigenvectors of the operator $\hat{T}(\hat{h})$, where

$$\hat{h} = \begin{pmatrix} \exp\{i\Delta^{1/2}t/2\} & 0 \\ 0 & \exp\{-i\Delta^{1/2}t/2\} \end{pmatrix}. \qquad (36)$$

4 Matrix elements of the irreps of $\hat{SU}(2)$ and isoLegendre polynomials

In this Section, we calculate matrix elements of the irreps $\hat{T}_l(\hat{u})$ of $\hat{SU}(2)$, and express the matrix elements $\hat{t}^l_{mn}(\hat{g})$ through the isoEuler angles ($\hat{\varphi}$, $\hat{\theta}$, $\hat{\psi}$) of the matrix $\hat{g}$.

The representations $\hat{T}_l(\hat{g})$ of $\hat{SL}(2,C)$ are given by

$$\hat{T}_l(\hat{g})\Delta\hat{\varphi}(\hat{x}) = (\beta x + \delta\Delta^{-1})^{2l}\Delta\hat{\varphi}\frac{\alpha x + \gamma\Delta^{-1}}{\beta x + \delta\Delta^{-1}}, \tag{37}$$

where $\hat{\varphi}(\hat{x})$ is polynomial of degree $2l$ on $\hat{x}$, and $\hat{g} \in \hat{SL}(2,C)$.

Using the isocanonical basis of Sec. 3 and the formula $a_{ij} = (e_j \hat{,} e_i)$, where $\{e_i\}$ is orthonormalized basis, we write down the matrix element,

$$\hat{t}^l_{mn}(\hat{g}) = (\hat{T}_l(\hat{g})\hat{\psi}_n \hat{,} \hat{\psi}_m) = \frac{\left(\hat{T}_l(\hat{g})\hat{x}^{l-n} \hat{,} \hat{x}^{l-m}\right)}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\Delta^{3/2}}, \tag{38}$$

where

$$\hat{\psi}_n(\hat{x}) = \frac{x^{l-n}\Delta^{3/2}}{\sqrt{(l-n)!(l+n)!}}, \tag{39}$$

$$-l \le n \le l, \quad \hat{(}l-n)! = \Delta^s(l-n)(l-n+1)\ldots, \qquad s = 1, 2, \ldots$$

On the other hand,

$$\hat{T}(\hat{g})x^{l-n} = (\alpha x + \gamma\Delta^{-1})^{l-n}\Delta(\beta x + \delta\Delta^{-1})^{l+n}, \tag{40}$$

so that (38) yields

$$\hat{t}^l_{mn} = \frac{\left((\alpha x + \gamma\Delta - 1)^{l-n}\Delta(\beta x + \delta\Delta^{-1})^{l+n} \hat{,} x^{l-m}\Delta^{-1}\right)}{\sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}}\Delta - 3/2. \tag{41}$$

Taking into account that $(\hat{x}^{l-k}, \hat{x}^{l-m}) = 0$ at $k \neq m$ and $(\hat{x}^{l-m}, \hat{x}^{l-m}) = (l-m)!(l+m)!\Delta^{2s+1}$, we have finally from (41)

$$\hat{t}^l_{mn}(\hat{g}) = \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\Delta^{2+l}$$

$$\times \sum_{j=M}^{N} \hat{C}^{l-m-j}_{l-n}\hat{C}^{j}_{l+n}\alpha^{l-m-j}\beta^j\gamma^{m+j-n}\delta^{l+n-j}$$

$$= \sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\Delta^{2\alpha-4s}\alpha^{l-m}\gamma^{m-n}\delta^{l+n}$$

$$\times \sum_{j=M}^{N} \frac{\Delta^{-4s-1}}{j!(l-m-j)!(l+n-j)!(m-n+j)!}\left(\frac{\hat{\beta}\hat{\gamma}}{\hat{\alpha}\hat{\delta}}\right)^j$$

$$= \sqrt{(l-m)!(l+m)!(l-n)!(l+n)!}\alpha^{l-m}\gamma^{m-n}\delta^{l+n}$$
$$\times \sum_{j=M}^{N} \frac{(\Delta^9/\Delta^{11s})^{1/2}\Delta^j}{j!(l-m-j)!(l+n-j)!(m-n+j)!}\Big(\frac{\beta\gamma}{\alpha\delta}\Big)^j, \qquad (42)$$

where $M = \max(0, n-m)$, $N = \min(l-m, l+n)$. We should to note that the matrix element (42) in fact does not depend on β because of isounimodularity of $\hat{g}$ implying $\beta\gamma = \alpha\delta - \Delta^{-1}$.

We are in a position to express $\hat{t}^l_{mn}(\hat{g})$ in terms of the isoEuler angles. Due to (32),

$$\hat{T}_l[\hat{g}(\hat{\varphi}, \hat{\theta}, \hat{\psi})] = \hat{T}_l[(\hat{g}(\hat{\varphi}, 0, 0)]\Delta\hat{T}^l[\hat{g}(0, \hat{\theta}, 0)]\Delta\hat{T}[\hat{g}(0, 0, \hat{\psi})], \qquad (43)$$

so that finding the general matrix $\hat{T}_l(\hat{g})$ reduces to finding of the matrices $\hat{T}_l[\hat{g}(\hat{\varphi}, 0, 0)]$, $\hat{T}_l[\hat{g}(0, \hat{\theta}, 0)]$, and $\hat{T}_l[\hat{g}(0, 0, \hat{\psi})]$.

The matrix $\hat{g}(\hat{\varphi}, 0, 0)$ is diagonal,

$$\hat{g}(\hat{\varphi}, 0, 0) = \begin{pmatrix} \exp\{i\Delta^{3/2}\varphi/2\} & 0 \\ 0 & \exp\{-i\Delta^{3/2}\varphi/2\} \end{pmatrix}. \qquad (44)$$

For this matrix, we have (see, for example, Ref.[4] for the ordinary case)

$$\hat{T}_l[\hat{g}(\hat{\varphi}, 0, 0)]\Delta 3/2 x^{l-n} = \exp -i\Delta^2 n\varphi\Delta^{1/2}\Delta^{3/2}x^{l-n}. \qquad (45)$$

Hence, the matrix of the operator $\hat{T}_l[\hat{g}(\hat{\varphi}, 0, 0)]$ is diagonal too, with the nonzero elements being $\exp[-i\Delta^{5/2}\phi]$, $-l \le n \le l$. The matrix of the operator $\hat{T}_l[\hat{g}(0, 0, \hat{\psi})]$ has similar form.

Let us denote matrix element of the operator $\hat{T}_l[\hat{g}(0, \hat{\theta}, 0)]$ as $\hat{t}^l_{mn}(\hat{\theta})$. Then, according to diagonality of the matrices of the operators $\hat{T}_l[\hat{g}(\hat{\varphi}, 0, 0)]$ and $\hat{T}_l[\hat{g}(0, 0, \hat{\psi})]$, we obtain

$$\hat{t}^l_{mn} = \hat{t}^l_{mn}[\hat{g}(\hat{\varphi}, 0, 0)]\Delta\hat{t}^l_{mn}(\hat{\theta})\Delta\hat{t}^l_{nn}[\hat{g}(0, 0, \hat{\theta})\exp\{-i\Delta^2(m\varphi + n\psi)\}\Delta\hat{t}^l_{mn}(\theta). \qquad (46)$$

It remains to obtain $\hat{t}^l_{mn}(\hat{\theta})$. The matrix $\hat{g}(0, \hat{\theta}, 0)$ has the form

$$\hat{g}(0, \hat{\theta}, 0) = \begin{pmatrix} g_{11}^{-1/2}\cos\hat{\theta}/2 & i\Delta g_{22}^{-1/2}\sin\hat{\theta}/2 \\ i\Delta g_{22}^{-1/2}\sin\hat{\theta}/2 & g_{11}^{-1/2}\cos\hat{\theta}/2 \end{pmatrix}, \qquad (47)$$

where $0 \le Re\theta < \pi$.

In the same manner as in Ref.[4] we then have

$$\hat{t}^l_{mn}(\hat{\theta}) = i^{-m-n}\Delta^{5-3s+2j}\sqrt{\frac{(l-m)!(l-n)!}{(l+m)!(l+n)!}}\Big(\frac{g_{11}}{g_{22}}\Big)^{1/2}\text{cotan}^{m+n}[\theta\Delta^{1/2}/2]$$

$$\times \sum_{j=max(m,n)}^{l} \frac{(l+j)!i^{2j}}{(l-j)!(j-m)!(j-n)!} g_{22}^{-1/2} \sin[\theta\Delta^{1/2}/2]. \tag{48}$$

Parameter $\hat{\theta}$ varies within the range $0 \leq Re\hat{\theta} < \pi$ so that, in this range, different values of $\hat{\theta}$ correspond to different values of $\hat{z} = g_{11}^{-1/2} \cos[\theta\Delta^{1/2}]$. So, $\hat{t}^l_{mn}(\hat{\theta})$ can be viewed as a function on isocos$\hat{\theta}$. In accordance to this, we put

$$\hat{t}^l_{mn}(\hat{\theta}) = \hat{P}^l_{mn}(g_{11}^{-1/2} \cos[\theta\Delta^{1/2}]. \tag{49}$$

Then, (46) can be rewritten as

$$\hat{t}^l_{mn}(\hat{\theta}) = \exp -i\Delta^{1/2}(m\varphi + n\psi)\Delta\hat{P}^l_{mn}(\hat{z}). \tag{50}$$

With the use of (50), Eq.(48) leads to the following definition of the isoLegendre polynomials:

$$\hat{P}^l_{mn} = i^{-m-n}\Delta^{5-2s+3j+(m+n)/2}\sqrt{\frac{(l-m)!(l-n)!}{(l+m)!(l+n)!}}\left(\frac{\Delta^{-1}+\hat{z}}{\Delta^{-1}-\hat{z}}\right)^{(m+n)/2}$$

$$\times \sum_{j=max(m,n)}^{l} \frac{(l+j)!i^{2j}}{(l-j)!(j-m)!(j-n)!}\left(\frac{\Delta^{-1}-\hat{z}}{2}\right)^{j}. \tag{51}$$

The factor $((\Delta^{-1}+\hat{z})/(\Delta^{-1}-\hat{z}))^{(m+n)/2}$ is twovalued since m and n are both integer or half-integer. Single valued definition in (51) comes when taking into account that $0 \leq Re\hat{\theta} < \pi$ and $\hat{z}$ maps this range to the plane $\hat{z}$ cutted along the real axis, $(-\infty;-1)$ and $(1;\infty)$. In the cutted plane the factor is single valued.

5 Basic properties of the isoLegendre polynomials

In this Section, we study the basic relations obeyed by the isoLegendre polynomials.

5.1 Symmetry relations

We will show that $\hat{P}^l_{mn}(\hat{z})$ is invariant under the changing of signs of the indeces m and n. For this purpose, we use the relation

$$\hat{g}(\pi)\hat{g}(\hat{\theta}) = \hat{g}(\hat{\theta})\hat{g}(\pi), \tag{52}$$

where we have denoted for brevity

$$\hat{g}(\hat{\theta}) = \begin{pmatrix} g_{11}^{-1/2} \cos[t\Delta^{1/2}/2] & i\Delta g_{22}^{-1/2} \sin[t\Delta^{1/2}/2] \\ i\Delta(g_{22})^{-1/2} \sin[t\Delta^{1/2}/2] & g_{11}^{-1/2} \cos[t\Delta^{1/2}/2] \end{pmatrix}. \quad (53)$$

From (52)-(53) it follows that

$$\hat{T}^l(\pi)\hat{T}^l(\hat{\theta}) = \hat{T}^l(\hat{\theta})\hat{T}^l(\pi). \quad (54)$$

Recall that the matrix elements of $\hat{T}_l(\hat{\theta})$ are just $\hat{P}^l_{mn}(\hat{z})$. Also, it is known that $\hat{t}^l_{mn}(\pi) = 0$ at $m+n \neq 0$, and $\hat{t}^l_{m,-m}(\pi) = i^{2\Delta l}$. Replacing the operators in (54) by their matrix elements we obtain

$$\hat{P}^l_{m,-n}(\hat{z}) = \hat{P}^l_{-m,n}(\hat{z}), \quad (55)$$

from which we have

$$\hat{P}^l_{mn}(\hat{z}) = \hat{P}^l_{-m,-n}(\hat{z}). \quad (56)$$

According to the explicit representation (51), we then also obtain

$$\hat{P}^l_{mn}(\hat{z}) = \hat{P}^l_{nm}(\hat{z}). \quad (57)$$

The relations (55), (56) and (57) are the basic symmetry relations for the isoLegendre polunomials.

The relations (56) and (57) means, particularly, that $\hat{P}^l_{mn}(\hat{z})$ depends on m and n through the combinations $|m+n|$ and $|m-n|$.

Also, it is straightforward to verify that the following relation holds,

$$\hat{P}^l_{mn}(\hat{z}) = i^{2\Delta(l-m-n)}\Delta\hat{P}^l_{m,-n}(\hat{z}). \quad (58)$$

5.2 Counter relations

The function $\hat{P}^l_{mn}(\hat{z})$ is defined in complex plane cutted along the lines $(-\infty;-1)$ and $(1;\infty)$. On the upper and lower neighbours of these lines $\hat{P}^l_{mn}(\hat{z})$ takes different values. From (51) it follows that for $\hat{z} > 1$ we have

$$\hat{P}^l_{mn}(\hat{z}+i0) = -\frac{1}{\Delta^{m-n}}\Delta\hat{P}^l_{mn}(\hat{z}-i0). \quad (59)$$

Similarly, for $\hat{z} < -1$,

$$\hat{P}^l_{mn}(\hat{z}+i0) = -\frac{1}{\Delta^{m+n}}\Delta\hat{P}^l_{mn}(\hat{z}-i0). \quad (60)$$

5.3 Relation to classical orthogonal polynomials

In Sec. 4, we have defined the isoLegendre function $\hat{P}^l_{mn}(\hat{z})$, and obtained one of the representaions of it. Now, we relate $\hat{P}^l_{mn}(\hat{z})$ to some of classical orthogonal polynomials - isoJacobi, adjoint isoLegendre, and isoLegendre polynomials.

This relations allows, particularly, to establish properties of the polynomials by the use of the properties of the isoLegendre function.

5.3.1 IsoJacobi polynomials

IsoJacobi polynomials are defined by

$$\hat{P}^{\hat{\alpha},\hat{\beta}}_k(\hat{z}) = \frac{(-\Delta^{-1})^k}{2^k k!}(1-z)^{-\alpha}\Delta^{1/2}(1+z)^{-\beta}\Delta^{1/2}$$

$$\times\frac{d^k}{d\hat{z}^k}[(1-z^2)^k(1+z)^{\alpha}\Delta^{1/2}(1+z)^{\beta}\Delta^{1/2}]\Delta^{5-k-s}. \tag{61}$$

Comparing (61) with the following representation of the isoLegendre function,

$$\hat{P}^l_{mn}(\hat{z}) = \frac{\Delta^{n-m-l}}{2^l}\sqrt{\frac{(l+m)!}{(l-n)!(l+n)(l-n)!}}$$

$$\times(1+z)^{-(m+n)/2}(1-z)^{(n-m)/2}\frac{d^{l-m}}{d\hat{z}^{l-m}}[(1-z)^{l-n}(1+z)^{l+n}]\Delta^{2-s+2l}, \tag{62}$$

we obtain

$$\hat{P}^{\hat{\alpha},\hat{\beta}}_k(\hat{z}) = 2^{m-n}\sqrt{(l-n)!(l+n)!(l-m)!(l+m)!}$$

$$\times(1-z)^{(n-m)/2}(1-z)^{-(n+m)/2}\hat{P}^l_{mn}(\hat{z})\Delta^{2+2m-n}, \tag{63}$$

where

$$l = k + \frac{\hat{\alpha}+\hat{\beta}}{2}, \quad m = \frac{\hat{\alpha}+\hat{\beta}}{2}, \quad n = \frac{\hat{\beta}-\hat{\alpha}}{2}. \tag{64}$$

From (30) we see that $\hat{\alpha} = m-n$ and $\hat{\beta} = m+n$ are integer numbers. Thus, $\hat{P}^l_{mn}(\hat{z})$ is reduced to isoJacobi polymomials, for which $\hat{\alpha}$ and $\hat{\beta}$ are integer numbers.

5.3.2 isoLegendre polynomials

IsoLegendre polynomials are defined by

$$\hat{P}_l(\hat{z}) = -\frac{\Delta^{-l}}{2^l l!}\frac{d^l}{d\hat{z}^l}[(1-z^2)^l]\Delta^{-(s+l)}. \tag{65}$$

This implies that $\hat{P}_l(\hat{z}) = \hat{P}_l^{00}(\hat{z})$. Comparing (65) and (62) we obtain

$$\hat{P}_l(\hat{z}) = \hat{P}^l_{00}(\hat{z}). \tag{66}$$

5.3.3 Adjoint isoLegendre functions

The adjoint isoLegendre function $\hat{P}_l^m(\hat{z})$, where $m \geq 0$ (l, m are integer), is defined by

$$\hat{P}_l^m(\hat{z}) = \frac{(-\Delta^{-1})^l}{2^{l+m}l!}(1-z^2)^{m/2}\frac{d^l}{d\hat{z}^l}[(1-z^2)^l]\Delta^{-(s+2l+m/2)}, \tag{67}$$

that is

$$\hat{P}_l^m(\hat{z}) = \frac{2^m(l+m)!}{l!}(1-z^2)^{m/2}\hat{P}_{l+m}^{-m,-m}(\hat{z})\Delta^{m/2}. \tag{68}$$

Comparing (68) with (62) leads to the following relation:

$$\hat{P}_l^m(\hat{z}) = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{-m0}(\hat{z})\Delta^{2-m}. \tag{69}$$

Let us rewrite (69) by taking into account (56),

$$\hat{P}_l^m(\hat{z}) = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{m0}(\hat{z})\Delta^{2-m}, \quad m \geq 0. \tag{70}$$

6 Functional relations for isoLegendre functions

In this Section, we derive basic theorems of composition and multiplication of $\hat{P}^l_{mn}(\hat{z})$, and the condition of its orthogonality.

6.1 Theorem of composition

Many important properties of $\hat{P}^l_{mn}(\hat{z})$ are related to the theorem of composition. To derive the rule, let us use the relation

$$\hat{T}^l(\hat{g}_1\Delta\hat{g}_2) = \hat{T}^l(\hat{g}_1)\Delta\hat{T}^l(\hat{g}_2), \tag{71}$$

from which it follows that

$$\hat{t}^l(\hat{g}_1 \Delta \hat{g}_2) = \sum_{k=-l}^{l} \hat{t}^l(\hat{g}_1)\Delta \hat{t}^l(\hat{g}_2), \tag{72}$$

and it can be rewritten as

$$\hat{t}^l_{mn}(\hat{g}_1 \Delta \hat{g}_2) = \exp\{-i\Delta^{1/2}(m\varphi + n\psi)\}\hat{P}^l_{mn}(\hat{z}). \tag{73}$$

For

$$\hat{t}^l_{mn}(\hat{g}_1 = \hat{P}^l_{mk}(\hat{z}), \quad \hat{t}^l_{kn}(\hat{g}_2 = \exp\{-i\Delta^{3/2}\varphi_2\}\Delta\hat{P}^l_{kn}(\hat{z}), \tag{74}$$

where $\hat{\varphi}$, $\hat{\theta}$, $\hat{\psi}$ are isoEuler angles of the matrix $\hat{g}_1 \Delta \hat{g}_2$. These angles are expressed through the angles $\hat{\theta}_{1,2}$, $\hat{\varphi}_2$ due to the following formulas:

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$

$$-g_{22}^{-1/2}\sin[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\cos[\varphi_2\Delta^{1/2}], \tag{75}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]\Delta\cos[\psi_2]\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}$$

$$+\frac{ig_{22}^{-1/2}\ sin[\theta_2\Delta^{1/2}]\Delta\sin[\varphi_2+\psi_1]\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{76}$$

$$\exp\{i(\hat{\varphi}+\hat{\psi})/2\} = \frac{g_{11}^{-1}\cos[\hat{\theta}_1/2]\cos[\hat{\theta}_2/2]\Delta\exp\{i\hat{\varphi}_2/2\}\Delta\exp\{\hat{\varphi}_2/2\}}{g_{11}^{-1/2}\cos[\hat{\psi}/2]},$$

$$-\frac{g_{22}^{-1/2}\sin[\hat{\psi}_1/2]\Delta\sin[\hat{\theta}_2/2]\Delta\exp\{-i\hat{\varphi}_2/2\}}{g_{11}^{-1/2}\cos[\hat{\psi}/2]}, \tag{77}$$

where $0 \leq Re\,\hat{\theta} < \pi$, $0 \leq Re\,\hat{\varphi} < 2\pi$, and $-2\pi \leq Re\,\hat{\psi} < 2\pi$.

Inserting equations (73) and (74) into (72), we obtain

$$\exp\{-i\Delta^{1/2}(m\varphi + n\psi)\}\hat{P}^l_{mn}(\hat{z})$$

$$= \sum_{k=-l}^{l} \exp\{-i\Delta^{3/2}\varphi_2\}\hat{P}^l_{mk}(\hat{z}_1)\hat{P}^l_{kn}(\hat{z}_2)\Delta^2. \tag{78}$$

(a) Let $\hat{\varphi}_2 = 0$, then if $Re(\hat{\theta}_1 + \hat{\theta}_2) < \pi \Rightarrow \hat{\theta} = \hat{\theta}_1 + \hat{\theta}_2$ and $\hat{\varphi} = \hat{\psi} = 0$. Accordingly, (78) takes the form

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2) = \sum_{k=-l}^{l} \hat{P}^l_{mk}(\hat{z}_1)\Delta\hat{P}^l_{kn}(\hat{z}_2) \Leftrightarrow$$

$$\hat{P}^l_{mn}[g_{11}^{-1/2}\cos[(\hat{\theta}_1 + \hat{\theta}_2)\Delta^{1/2}] \tag{79}$$

$$= \sum_{k=-l}^{l} \hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2]\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]).$$

(b) Let $\varphi_2 = 0$, then if $Re(\hat{\theta}_1 + \hat{\theta}_2) > \pi \Rightarrow \hat{\theta} = 2\pi - \hat{\theta}_1 - \hat{\theta}_2$, $\hat{\varphi} = \hat{\psi} = \pi$. Therefore,

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2) = -\Delta^{2-m-n}$$

$$\sum_{k=-l}^{l} \hat{P}^l_{mk}(g_{11}^{-1/2}cos[\theta_1\Delta^{1/2}2])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]). \tag{80}$$

(c) Let $\varphi_2 = \pi$, then if $Re\,\hat{\theta}_1 \geq Re\,\hat{\theta}_2$, $\Rightarrow \hat{\theta} = \hat{\theta}_1 - \hat{\theta}_2$, $\hat{\varphi} = 0$, $\hat{\psi} = \pi$. Therefore,

$$\hat{P}^l_{mn}(\hat{z}_1 + \hat{z}_2)$$

$$= \sum_{k=-l}^{l} (-\Delta^{2-n-k})\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]). \tag{81}$$

(d) In particular, at $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}$, we have

$$\sum_{k=-l}^{l} (-\Delta^{1-n-k})\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}2])$$

$$\times\Delta\hat{P}^l kn(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}2]) = \delta^{mn}. \tag{82}$$

(e) At $\hat{\varphi} = \frac{\pi}{2}$, the formulas (75)-(77) take the following forms:

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}], \tag{83}$$

$$\exp\{i\Delta^{1/2}\varphi\} = \frac{\sin[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}] + i\Delta\sin[\theta_2\Delta^{1/2}]}{\sin[\theta\Delta^{1/2}]}, \tag{84}$$

$$\exp\{i\Delta^{1/2}(\varphi + \psi)/2\} = \frac{\cos[(\theta_1 + \theta_2)\Delta^{1/2}/2] + i\Delta\cos[(\theta_1 - \theta_2)\Delta^{1/2}/2]}{\cos[\theta_2\Delta^{1/2}]}, \tag{85}$$

respectively.

Instead of (84) and (85), it is more convenient to define

$$gg_{22}^{-1/2}tan[\varphi\Delta^{1/2}] = \frac{\sin[\{\theta_2\Delta^{1/2}]}{\sin}[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}], \tag{86}$$

$$g_{22}^{-1/2}\tan[\psi\Delta^{1/2}] = \frac{\sin[\{\theta_1\Delta^{1/2}]}{\cos[\theta_1\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]}. \tag{87}$$

Then

$$\exp\{-i\Delta^{1/2}(m\varphi+n\psi)\}\Delta\hat{P}^l_{mn}(g_{11}^{-1/2}cos[\theta_2\Delta^{1/2}])$$

$$= i^{-k}\sum_{k=-l}^{l}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \tag{88}$$

6.1.1 Theorem of composition for isoLegendre polynomials

Consider particular cases of the function $\hat{P}^l_{mn}(\hat{z})$, namely, the IsoLegendre polynomials and adjoint isoLegendre polynomials. The polynomials are defined due to

$$\hat{P}_l(\hat{z}) = \hat{P}^l_{00}(\hat{z}), \quad \hat{P}^m_l = i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}^l_{-m0}(\hat{z})\Delta^{2-m}. \tag{89}$$

Taking into account the formulas from Sec 6.1 and using (89) we get

$$\exp\{i\Delta^{3/2}m\varphi\}\hat{P}^l_m(g_{11}^{-1/2}\cos[\theta^{1/2}_\Delta])$$

$$= i^m\sqrt{\frac{(l+m)!}{(l-m)!}}\sum_{k=-l}^{l}i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}} \tag{90}$$

$$\times\exp\{-i\Delta^{3/2}m\varphi_2\}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^k_l(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^{5+m-k},$$

where $\hat{\varphi}$, $\hat{\theta}$, $\hat{\varphi}_2$, and $\hat{\theta}_1$, $\hat{\theta}_2$ are related to each other as in Sec. 6.1.

If we put $m = n = 0$, we obtain, particularly,

$$\hat{P}^l(g_{11}^{-1}\cos[\theta_1\Delta^{1/2}])\Delta\cos[\theta_2\Delta^{1/2}]$$

$$-g_{22}^{-1}\sin([\theta_1\Delta^{1/2}]\sin([\theta_2\Delta^{1/2}]g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]) \tag{91}$$

$$= (-1)\sum_{k=-l}^{l}i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}}\exp\{-i\Delta^{3/2}m\varphi_2\}$$

$$\times \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^4.$$

Due to the symmetry $\hat{P}^l_{m0}(\hat{z}) = \hat{P}^l_{-m0}(\hat{z})$ and (89), the relation (91) can be reduced:

$$\hat{P}_l^{-m}(\hat{z}) = -1\sqrt{\frac{(l+m)!}{(l-m)!}}\hat{P}_l^m(\hat{z})\Delta^{2-m}. \tag{92}$$

Thus, from (91) it follows that the polynomials - $\hat{P}_l(\hat{z})$ obey the following *theorem of composition*:

$$\hat{P}^l(g_{11}^{-1}\cos[\theta_1\Delta^{1/2}])\Delta\cos[\theta_2\Delta^{1/2}]$$

$$-g_{22}^{-1}\sin([\theta_1\Delta^{1/2}]\sin[\varphi_2\Delta^{1/2}]g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]\Delta\sin[\theta_2\Delta^{1/2}]) \tag{93}$$

$$= (-1)\sum_{k=-l}^{l} i^{-k}\sqrt{\frac{(l-k)!}{(l+k)!}}\exp\{-i\Delta^{3/2}m\varphi_2\}$$

$$\times \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^3.$$

6.2 Multiplication rules

Let in the composition rule

$$\exp\{-i\Delta^{3/2}(m\varphi+n\psi)\}\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])$$

$$= \exp\{-i\Delta^{3/2}k\varphi_2\}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^2. \tag{94}$$

If φ_2 is a real angle, then this formula can be viewed as a Fourier expansion of the function

$$\exp\{-i\Delta^{3/2}(m\varphi+n\psi)\}\Delta\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]).$$

Therefore,

$$\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

$$= \frac{\Delta^3}{2\pi}\int_{-\pi}^{\pi}\exp\{-i\Delta^{3/2}(k\varphi_2-m\varphi-n\psi)\}\hat{P}^l_{mn}(g_{11}^{-1/2})d(\varphi_2\Delta^{1/2}). \tag{95}$$

Putting $m = n = 0$ in this formula, we get

$$\frac{\Delta^3}{2\pi}\int_{-\pi}^{\pi}\exp\{-i\Delta^{3/2}k\varphi_2\}\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2})d(\varphi_2\Delta^{1/2})$$

$$= \sqrt{\frac{(l-k)!}{(l+k)!}} \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])\Delta^{3-k}. \tag{96}$$

Since $\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])$ is an even function in respect to φ_2, the above equality can be rewritten

$$\frac{\Delta^2}{\pi}\int\limits_{-\pi}^{\pi} \hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])g_{11}^{-1/2}\cos[k\varphi_2\Delta^{1/2}]d(\varphi_2\Delta^{1/2})$$

$$= \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \tag{97}$$

If we now let additionally $k = 0$, we obtin the further reduction

$$\frac{\Delta}{\pi}\int\limits_{-\pi}^{\pi} \hat{P}^l_{mn}(g_{11}^{-1/2} cos[\theta\Delta^{1/2})d(\varphi_2\Delta^{1/2})$$

$$= \hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \tag{98}$$

Let us rewrite eq. (98) in a more convenient form. Assuming θ_1, θ_2, φ_2 to be real numbers such that $0 \le \theta_1 < \pi$ and $0 \le \theta_1 + \theta_2 < \pi$, we redefine the variable

$$\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$$
$$-g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\sin[\theta_2\Delta^{1/2}]\cos[\varphi_2\Delta^{1/2}]\Delta^2. \tag{99}$$

Introduce the notation

$$\hat{T}_n(\hat{x}) = g_{11}^{-1/2}\cos[n\Delta g_{11}^{-1/2}\text{arccos}\hat{x}].$$

This function defines *Chebyshev-I polynomial*. From the last equation it follows that

$$g_{11}^{-1/2}\cos[k\Delta^{3/2}\varphi_2]$$
$$= \hat{T}_k\frac{g_{11}^{-1}(\cos[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}] - \cos[\theta\Delta^{1/2}])}{g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\Delta g_{22}^{-1}\sin[\theta_2\Delta^{1/2}]}. \tag{100}$$

In turn, from the condition (100) it follows

$$d\hat{\varphi}_2 =$$

$$\frac{g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]\Delta d\hat{\theta}}{\sqrt{g_{11}^{-1/2}(\cos[\theta\Delta^{1/2}] - \cos[(\theta_1+\theta_2)\Delta^{1/2}])\Delta g_{11}^{-1/2}(\cos[(\theta_1-\theta_2) - \cos[\theta\Delta^{1/2}]}}. \tag{101}$$

Since when varying $\hat{\varphi}_2$ from $\hat{\theta}$ to π the variable $\hat{\theta}$ varies in the range from $|\hat{\theta}_1+\hat{\theta}_2|$ to $|\hat{\theta}_1-\hat{\theta}_2|$, the above made redefinition transfroms the integral to the form

$$\frac{\Delta^2}{2}\int\limits_{|\theta_1+\theta_2|}^{\theta_1+\theta_2}\hat{P}_l^k(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}])$$

$$\times\hat{T}_k\frac{g_{11}^{-1}\cos[\theta_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[\theta_2\Delta^{1/2}]-g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]}{g_{22}^{-1}\sin[\theta_1\Delta^{1/2}]\Delta g_{22}^{-1}\sin[\theta_2\Delta^{1/2}]}$$

$$\times\frac{g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]\Delta d(\hat{\theta}\Delta^{1/2})}{g_{11}^{-1/2}(\cos[\theta\Delta^{1/2}]-\cos[(\theta_1+\theta_2)\Delta^{1/2}])\Delta g_{11}^{-1/2}(\cos[(\theta_1-\theta_2)\Delta^{1/2}]\cos[\theta\Delta^{1/2}])}$$

$$=\hat{P}_l^k(g_{11}^{-1/2}cos[\theta_1\Delta^{1/2}])\Delta\hat{P}_l^{-k}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}]). \qquad (102)$$

The expression in the denominator has a simple geometrical meaning: it is equal to the square of the spherical triangle with the sides $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}$, divided to $4\pi^2$.

6.3 Orthogonality relations

In this Section, we apply theorems of orthogonality and completeness of the system of matrix elements of pairwise nonequivalent irreducible isounitary representations of compact group to the group $\hat{SU}(2)$. Since dimension of the representation $\hat{T}_l(\hat{u})$ of the group $\hat{SU}(2)$ is $2l+1$, the functions $\sqrt{2l+1}\Delta\hat{t}^l_{mn}(\hat{u})$ form complete orthogonal normalized system in respect to invariant measure $d\hat{u}$ on this group. In other words, the functions $\hat{t}^l_{mn}(\hat{u})$ fulfill the relations

$$\int\limits_{\hat{SU}(2)}\hat{t}^l_{mn}(\hat{u})\Delta^2\hat{t}^{*k}_{pq}(\hat{u})d\hat{u}=\frac{\Delta^2}{(2l+1)}\delta_{lk}\delta_{mp}\delta_{nq}. \qquad (103)$$

Inserting expression for the matrix elements

$$\hat{t}^l_{mn}(\hat{\varphi},\hat{\theta},\hat{\psi})=\exp\{-i\Delta^{3/2}(m\varphi+n\psi)\}\hat{P}^l(g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]) \qquad (104)$$

into (103) and using the fact that the measure $d\hat{u}$ on the group $\hat{SU}(2)$ is given by

$$d\hat{u}=\frac{\Delta^4}{16\pi}gg_{22}^{-1/2}sin[\theta\Delta^{1/2}]d\hat{\varphi}d\hat{\theta}d\hat{\psi}, \qquad (105)$$

we turn to the following specific cases.

(a) If $l \neq k$ or $m \neq p$ or $n \neq q$, then

$$\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi} \hat{P}^{l}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^{*k}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])g_{22}^{-1/2}\sin[\theta\Delta^{1/2}]$$

$$\times\Delta^6\exp\{i\Delta^{3/2}(p-m)\varphi\}\exp\{i\Delta^{3/2}(q-n)\psi\}\psi d(\theta\Delta^{1/2})d(\varphi\Delta^{1/2})d(\psi\Delta^{1/2}). \tag{106}$$

(b) Let $p = m$ and $q = n$, then, at $l \neq k$,

$$\int_{0}^{\pi} \hat{P}^{l}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])\hat{P}^{*k}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])g_{11}^{-1/2}\cos[\hat{\theta}]\Delta^3 d(\theta\Delta^{1/2}) = 0. \tag{107}$$

Analogously, from (103) it follows

$$\int_{0}^{\pi} |\hat{P}^{l}_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])|^2 g_{11}^{-1/2}\sin[\theta\Delta^{1/2}]d(\theta\Delta^{1/2}) = \frac{2}{2l+1}. \tag{108}$$

Further, putting $\hat{x} = g_{11}^{-1/2}\cos[\theta\Delta^{1/2}]$ we get the orthogonality relations for $\hat{P}^{l}_{mn}(\hat{x})$:

$$\int_{-1}^{1} \hat{P}^{l}_{mn}(\hat{x})\hat{P}^{*k}_{mn}(\hat{x})d(\hat{x}) = \frac{2}{2l+1}\delta_{lk}. \tag{109}$$

7 Recurrency relations for isoLegendre functions

In this Section, we derive the formulas relating the functions $\hat{P}^{l}_{mn}(\hat{z})$, indeces of which are differ from each other by one, that is, recurrency relations, which can be viewed as an infinitesimal form of the theorem of composition. These relations are then follow from the composition rules at infinitesimal $\hat{\theta}_2$.

To obtain the reccurency rules, we diffirentiate the equation below on $\hat{\theta}_2$ and put $\hat{\theta}_2 = 0$:

$$\hat{P}^{l}_{mn}(\hat{z}) = \sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\times$$

$$\times\frac{\Delta^7}{2\pi}\int_{0}^{2\pi} d\varphi(g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{i\Delta^{1/2}}{2}$$

$$+ig_{22}^{-1/2}\sin[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{-i\Delta^{1/2}}{2})^{l-n}(ig_{22}^{-1/2}\sin[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{i\Delta^{1/2}}{2}$$

$$\times g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]\exp\frac{-i\Delta^{1/2}\varphi}{2})^{l-n}\exp i\Delta^{1/2}\varphi. \tag{110}$$

First, we find

$$\frac{d}{d\hat{\theta}}[\hat{P}^l_{mn}(g-11^{-1/2}\cos[\theta\Delta^{1/2}])]_{|\hat{\theta}=0}$$

$$=\frac{\Delta^5}{4\pi}\sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}} \tag{111}$$

$$\times\int_0^{2\pi} d\varphi(l-n)\exp -i\Delta^{3/2}(n+1)\varphi+(l+n)\exp -i\Delta^{3/2}(n-1)\varphi\exp\Delta^{3/2}m\varphi.$$

It is obvious that the r.h.s. of this equation is zero unless $m=n\pm 1$. At $m=n+1$, from (111) we get

$$\frac{d}{d\hat{\theta}}[\hat{P}^{n+1}_n\cos[\frac{\theta\Delta^{1/2}}{2}]_{|\hat{\theta}=0}=\frac{i}{2}\Delta^{3/2}\sqrt{(l-n)(l+n+1)}. \tag{112}$$

Similarly,

$$\frac{d}{d\hat{\theta}}[\hat{P}^{n-1}_n(g_{11}^{-1/2}\cos[\frac{\theta\Delta^{1/2}}{2}]_{\hat{\theta}=0}=\frac{i}{2}\Delta^{3/2}\sqrt{(l+n)(l-n+1)}. \tag{113}$$

Now, we are ready to derive the recurrency relations.

Using the factorization

$$\hat{P}^l_{mn}[g_{11}^{-1/2}\cos[(\theta_1+\theta_2)\Delta^{1/2}]]=\sum_{k=-l}^{l}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])$$
$$\Delta\hat{P}^l_{kn}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

obtained in Sec. GS6, putting $\hat{\theta}_2=0$ and replacing $g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$ by $\hat{z}$, we obtain the recurrency relation in the form

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(\hat{z})}{dz}=-\frac{i}{2}\Delta^{3/2}[\sqrt{(l-n)(l+n+1)} \tag{114}$$
$$\times\hat{P}^l_{m,n+1}(\hat{z})+\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z})].$$

To derive the second recurrency relation, we use the particular case of the theorem of composition which corresponds to $\hat{\varphi}_2 = \frac{\pi}{2}$. Namely, we differentiate the formula

$$\exp -i\Delta^{3/2}(m\varphi + n\psi)\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]) =$$

$$\sum_{k=-l}^{l} i^{-k}\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]$$

$$\Delta\hat{P}^l kn(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

and put $\theta_2 = 0$. After strightforward computations, we have

$$i\Delta^3\left[m\frac{d\hat{\varphi}}{d\hat{\theta}_2} + n\frac{d\hat{\psi}}{d\hat{\theta}_2}\right]|_{\hat{\theta}_2=0}$$

$$\times\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}] - \frac{d\hat{P}^l_{mk}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}])}{d\hat{\theta}_1}\frac{d\hat{\theta}}{\hat{\theta}_2}|_{\hat{\theta}_2=0} \tag{115}$$

$$-\frac{1}{2}\Delta^{3/2}\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(g_{11}^{-1/2}\cos[\theta_1\Delta^{1/2}]-$$

$$\sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(g_{11}^{-1/2}\cos[\theta_2\Delta^{1/2}])$$

It remains to find $d\hat{\varphi}/d\hat{\theta}_2$ and $d\hat{\psi}/d\hat{\theta}_2$. To this end, we differentiate the equality $\cos[\theta\Delta^{1/2}] = \cos[\theta_1\Delta^{1/2}]\Delta\cos[\theta_2\Delta^{1/2}]$. Since, at $\hat{\theta}_2 = 0$, we have $\hat{\theta} = \hat{\theta}_1$, $\hat{\varphi} = 0$, and $\hat{\psi} = \hat{\varphi}_2 = \frac{\pi}{2}$, it follows that $d\hat{\theta}/d\hat{\theta}_2|_{\hat{\theta}_2=0} = 0$. Similarly,

$$\frac{d\hat{\varphi}}{d\hat{\theta}_2}|_{\hat{\theta}_2=0} = \frac{g_{11}^{1/2}}{\cos[\theta_1\Delta^{1/2}]} \tag{116}$$

and

$$\frac{d\hat{\psi}}{d\hat{\theta}_2}|_{\hat{\theta}_2=0} = -(\frac{g_{11}}{g_{22}})^{1/2}ctan[\theta_1\Delta^{1/2}]$$

$$i\Delta\left[\frac{m-nz}{1-z^2}\right]\hat{P}^l_{mn}(\hat{z})$$

$$= \frac{1}{2}\left[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) - \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\right]. \tag{117}$$

From the reccurency relations obtained above it is straightforward to write down the following recurrency relations:

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} + \frac{nz-m}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z}) \tag{118}$$

and, analogously,

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} - \frac{nz-m}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n+1}(\hat{z}). \tag{119}$$

Due to the symmetry, we have from (118) and (119)

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} + \frac{mz-n}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l-m)(l+m+1)}\hat{P}^l_{m,n+1}(\hat{z}) \tag{120}$$

and

$$\sqrt{1-z^2}\frac{d\hat{P}^l_{mn}(z)}{d\hat{z}} - \frac{mz-n}{\sqrt{1-z^2}}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\sqrt{(l+m)(l-m+1)}\hat{P}^l_{m,n+1}(\hat{z}). \tag{121}$$

Adding (118) to (119), we obtain the recurrency relations for three $\hat{P}$'s:

$$2\left[\frac{n-mz}{1-z^2}\right]\hat{P}^l_{mn}(\hat{z}) = i\Delta^{3/2}\left[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) - \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\right], \tag{122}$$

$$\sqrt{1-z^2}\hat{P}^l_{mn}(\hat{z}) = -i\Delta^{5/2}\left[\sqrt{(l+n)(l-n+1)}\hat{P}^l_{m,n-1}(\hat{z}) + \sqrt{(l-n)(l+n+1)}\hat{P}^l_{m,n+1}(\hat{z})\right]. \tag{123}$$

Putting $m=0$ in (118) and (119), and using

$$\hat{P}^l_{0n}(\hat{z}) = i^{-n}\Delta^2\sqrt{\frac{(l-n)}{(l+n)}}\hat{P}^n_l(\hat{z}) \tag{124}$$

we obtain, finally, the recurrency rules for the adjoint isoLegendre polynomials,

$$\sqrt{1-z^2}\frac{d\hat{P}^n_l(\hat{z})}{d\hat{z}} + \Delta^2\frac{nz}{1-z^2}\hat{P}^n_l(\hat{z}) = -\hat{P}^{n+1}_l(\hat{z}) \tag{125}$$

and

$$\sqrt{1-z^2}\frac{d\hat{P}_l^n(\hat{z})}{d\hat{z}} - \Delta^2\frac{n}{1-z^2}\hat{P}_l^n(\hat{z}) = -\Delta^3(l+n)(l-n+1)\hat{P}_l^{n-1}(\hat{z}). \quad (126)$$

8 The group $\hat{QU}(2)$

In this Section, we consider the group $\hat{QU}(2)$ consisting of isounimodular isoquasiunitary matrices representations of which lead to isoJacobi and isoLegendre functions.

8.1 Definitions

The representations of $\hat{QU}(2)$ are in many ways similar to that of the group $\hat{SU}(2)$. However, in contrast to $\hat{SU}(2)$, the group $\hat{QU}(2)$ is not compact, thus having continuous series of isounitary representations.

Similarly to the description of the group $\hat{SU}(2)$, we describe the group $\hat{QU}(2)$ as a set of isounimodular isoquasiunitary 2×2 matrices

$$\hat{g}_0 = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}, \tag{127}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are given by (7), satisfying

$$\hat{g}_0 \Delta \hat{s} \Delta \hat{g}_0^* = \hat{s}, \tag{128}$$

where

$$\hat{s} = \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{pmatrix}, \quad \hat{g}_0^* = \begin{pmatrix} \bar{\hat{\alpha}} & \hat{\beta} \\ \bar{\hat{\beta}} & \hat{\alpha} \end{pmatrix} \quad \det \hat{g}_0 = 1, \quad |\hat{\alpha}|^{\hat{2}} - |\hat{\beta}|^{\hat{2}} = \Delta^{-1}. \tag{129}$$

8.2 Parametrizations

The matrices $\hat{g}_0$ above have been defined by the complex numbers $\hat{\alpha}$ and $\hat{\beta}$. However, in various aspects it is suitable to define them by the isoEuler angles. Constraints on the isoEuler angles following from the requirement that $\hat{g}_0 \in \hat{QU}(2)$ are

$$\begin{aligned} & g_{11}^{-1/2} \cos[\frac{\theta\Delta^{1/2}}{2}]\Delta \exp\{-i\Delta^{3/2}(\varphi + \psi)/2\} \\ &= g_{11}^{-1/2} \overline{\cos}[\theta\Delta^{1/2}/2]\Delta \exp\{-i\Delta^{3/2}(\bar{\varphi} + \bar{\psi})/2\} \end{aligned} \tag{130}$$

and

$$g_{11}^{-1/2} \sin[\frac{\theta\Delta^{1/2}}{2}]\Delta \exp\{i\Delta^{3/2}(\varphi - \psi)/2\}$$

$$= -g_{22}^{-1/2}\sin[\frac{\bar{\theta}\Delta^{1/2}}{2}]\Delta\exp\{i\Delta^{3/2}(\bar{\varphi}-\bar{\psi})/2\}, \tag{131}$$

which we rewrite in the following form:

$$\cos[\frac{i\theta\Delta^{1/2}}{2}] = \cos[\frac{\bar{\theta}\Delta^{1/2}}{2}]\exp\{i\Delta^{3/2}(\varphi-\bar{\varphi}+\psi-\bar{\psi})/2\} \tag{132}$$

and

$$\sin[\frac{i\theta\Delta^{1/2}}{2}] = -\sin[\frac{i\bar{\theta}\Delta^{1/2}}{2}]\exp\{i\Delta^{3/2}(\bar{\psi}-\varphi+\psi-\bar{\varphi})/2\}. \tag{133}$$

The angles $\varphi-\bar{\varphi}+\psi-\bar{\psi}$ and $\bar{\psi}-\varphi+\psi-\bar{\varphi}$ are real. So, if $\hat{g}_0 = \hat{g}_0(\hat{\varphi},\hat{\theta},\hat{\psi}) \in \hat{QU}(2)$ then $\cos[\theta\Delta^{1/2}/2]$ is an imaginary number, *i.e.* $\hat{\tau} = i\hat{\theta}$ is real.

Taking into account the constraints (132) and (133), we obtain the following ranges for the parameters:

$$0 \le \hat{\varphi} < 2\pi, \qquad 0 \le \hat{\tau} < \infty, \qquad -2\pi \le \hat{\psi} < 2\pi. \tag{134}$$

In terms of these parameters, the matrix $\hat{g}_0$ is

$$\hat{g}_0 = \begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}]\Delta e^{\frac{i\Delta^{3/2}(\varphi+\psi)}{2}} & -i\Delta^2 g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]e^{\frac{i\Delta^{3/2}(\varphi-\psi)}{2}} \\ -i\Delta^2 g_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]e^{\frac{i\Delta^{3/2}(\psi-\varphi)}{2}} & g_{11}^{-1/2}\cos[\frac{\psi\Delta^{1/2}}{2}]\Delta e^{\frac{-i\Delta^{3/2}(\varphi+\psi)}{2}} \end{pmatrix}. \tag{135}$$

Thus, we see that the group $\hat{QU}(2)$ is one of the real types of subgroups of $\hat{SL}(2,C)$. In the following, we use the parameters (134) instead of the isoEuler angles $(\hat{\varphi},\hat{\theta},\hat{\psi})$.

Let us find the transformation laws for these parameters under the multiplying of two elements of $\hat{QU}(2)$. We introduce the notation $\hat{g}_{01} = (0,\hat{\tau}_1,0)$ and $\hat{g}_{02} = (\hat{\varphi}_2,\hat{\tau}_2,0)$ so that $\hat{g}_{01}\Delta\hat{g}_{02} = (\hat{\varphi},\hat{\tau},\hat{\psi})$. Using the formulas (16)-(18) we find

$$\cos[i\tau\Delta^{1/2}] == \cos[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]g_{11}^{-1/2} \tag{136}$$

$$-\sin[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}],$$

$$\exp\{i\Delta^{1/2}(\varphi+\psi)/2\} = \Delta^2(\frac{g_{11}^{-1}\cos[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]\exp\{i\Delta^{1/2}\varphi_2/2\}}{\cos[i\tau\Delta^{1/2}]}$$

$$+\frac{g_{22}^{-1}\sin[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\exp\{-i\Delta^{1/2}\varphi_2/2\})}{\cos[i\tau\Delta^{1/2}]} \tag{137}$$

and

$$\exp\{i\Delta^{1/2}\varphi\} = \Delta^{-1/2}(\frac{\sin[i\tau_1\Delta^{1/2}]\Delta\cos[i\tau_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]} \tag{138}$$

$$+\frac{\Delta^{-1/2}\cos[i\tau_1\Delta^{1/2}]\Delta\sin[i\tau_2\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[\varphi_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}$$

$$-\frac{ig_{22}^{-1}\sin[i\tau_2\Delta^{1/2}]\Delta\sin[\varphi_2\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}).$$

It is easy to check that the element $\hat{g}_0(\hat{\varphi},\hat{\tau},\hat{\psi})$ is an inverse of $\hat{g}_0(\pi-\hat{\varphi},\hat{\tau},-\pi-\hat{\psi})$.

8.3 Relation to the group $\hat{SH}(3)$

Let us define the group $\hat{SH}(3)$ as the group of isolinear transformation of three dimensional isoEuclidean space $\hat{E}_3$ acting transitively on (iso)hyperboloids and (iso)conics. This transformation is an isohyperbolic one.

The relation between the groups $\hat{QU}(2)$ and $\hat{SH}(3)$ is similar to that between $\hat{SU}(2)$ and $\hat{S}0(3)$. Namely, to every point $\hat{x}(\hat{x}_1,\hat{x}_2,\hat{x}_3)\in\hat{E}_3$ we associate the quasiuntary matrix

$$\hat{h}_x = \begin{pmatrix} \hat{x}_1 & \hat{x}_2+i\hat{x}_3 \\ \hat{x}_2-i\hat{x}_3 & -\hat{x}_1 \end{pmatrix}. \tag{139}$$

Then,

$$\hat{T}(\hat{g}_0)\Delta\hat{h}_x = \hat{g}_0\Delta\hat{h}_x^*\hat{g}_0. \tag{140}$$

Accordingly,

$$\hat{T}(\hat{g}_0)\Delta\hat{h}_x = \begin{pmatrix} \Delta^{-1}y_1 & \Delta y_2+iy_3 \\ \Delta y_2-iy_3 & \Delta y_1 \end{pmatrix}, \tag{141}$$

where $\hat{x}=g_{11}^{1/2}x$, $\hat{y}=g_{22}^{1/2}y$, and $\hat{y}(\hat{y}_1,\hat{y}_2,\hat{y}_3)in\hat{E}_3$.

9 Irreps of $\hat{QU}(2)$

9.1 Description of the irreps

Denote $\hat{\chi}=(l,\varepsilon)$, where l is complex number and $\varepsilon=0,1/2$. With every $\hat{\chi}$ we associate the space $D_{\hat{\chi}}$ of functions $\hat{\varphi}(\hat{z})$ of complex variable $\hat{z}=\hat{x}+i\hat{y}$ such that:

(1) $\hat{\varphi}(\hat{z})$ is of C^∞ class on $\hat{x}$ and $\hat{y}$ at every point $\hat{z} = \hat{x} + i\hat{y}$ except for $\hat{z} = 0$;

(2) for any $a > 0$ the following equation is satisfied:

$$\hat{\varphi}(a\Delta\hat{z}) = a^{2\Delta}\Delta\hat{\varphi}(\hat{z}). \tag{142}$$

(3) $\hat{\varphi}(\hat{z})$ is an even (odd) function at $\varepsilon = 0(1/2)$,

$$\hat{\varphi}(-\hat{z}) = (-\Delta^{-1})^{2\varepsilon}\Delta\hat{\varphi}(\hat{z}). \tag{143}$$

For subsequent purposes, we realize the space $D_{\hat{\chi}}$ on a circle. Namely, with every function $\hat{\varphi}(\hat{z})$ we associate the function $\hat{f}$ such that, at $\varepsilon = 0$,

$$\hat{f}(\exp\{i\theta\Delta^{1/2}\}) = \hat{\varphi}(\exp\{i\theta\Delta^{1/2}\}) \tag{144}$$

and, at $\varepsilon = 1/2$,

$$\hat{f}(\exp\{i\theta\Delta^{1/2}\}) = \exp\{i\theta\Delta^{1/2}\}\Delta\hat{\varphi}(\exp\{i\theta\Delta^{1/2}\}). \tag{145}$$

Thus, the space $D_{\hat{\chi}}$ can be represented as the space D of functions on circle.

9.2 Representations $\hat{T}_{\hat{\chi}}(\hat{g}_0)$

To every element

$$\hat{g}_0 = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \bar{\hat{\beta}} & \bar{\hat{\alpha}} \end{pmatrix}$$

of the group $\hat{Q}U(2)$ we associate the operator in the space $D_{\hat{\chi}}$,

$$\hat{T}_{\hat{\chi}}(\hat{g}_0)\Delta\hat{\varphi}(\hat{z}) = \hat{\varphi}(\hat{\alpha}\hat{z} + \bar{\hat{\beta}}\hat{z}). \tag{146}$$

Clearly, function $\hat{T}_{\chi}(\hat{g}_0)\Delta\hat{\varphi}(\hat{z})$ has the same homogeneity degree as the function $\hat{\varphi}(\hat{z})$, and so the operator $\hat{T}_{\hat{\chi}}(\hat{g}_0)$ is an automorphism of the space $D_{\hat{\chi}}$. Also, it is easy to verify that

$$\hat{T}_{\chi}(\hat{g}_{01})\Delta\hat{T}_{\chi}(\hat{g}_{01}) = \hat{T}_{\chi}(\hat{g}_{01}\Delta\hat{g}_{02}). \tag{147}$$

Action of the operator $\hat{T}_{\hat{\chi}}(\hat{g}_0)$ can then be strightforwardly derived. Namely, for $\hat{\chi} = (l, 0)$ we have

$$\hat{T}_{\hat{\chi}}(\hat{g})\Delta\hat{f}(\exp\{i\theta\Delta^{1/2}\})$$

$$= |\hat\beta\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat\alpha}|^{2l}\Delta^{2l+1}\hat f\left(\frac{\hat\alpha\exp\{i\theta\Delta^{1/2}\} + \bar{\hat\beta}}{\hat\beta\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat\alpha}}\right), \qquad (148)$$

and for $\hat\chi = (l, 1/2)$

$$\hat T_{\hat\chi}(\hat g)\Delta\hat f(\exp\{i\theta\Delta^{1/2}\})$$

$$= |\hat\beta\Delta\exp\{i\theta\Delta^{1/2}\}+\bar{\hat\alpha}|^{2l-1}\Delta^{2l+1}(\hat\beta\Delta\exp\{i\theta\Delta^{1/2}\}+\bar{\hat\alpha})\hat f\left(\frac{\hat\alpha\exp\{i\theta\Delta^{1/2}\} + \bar{\hat\beta}}{\hat\beta\Delta\exp\{i\theta\Delta^{1/2}\} + \bar{\hat\alpha}}\right). \qquad (149)$$

10 Matrix elements of the irreps of $\hat{QU}(2)$ and iso-Jacobi functions

10.1 The matrix elements

Let us choose the basis $\exp\{-im\theta\Delta^{3/2}\}$ in space $\hat D_{\hat\chi}$, and define the matrix elemets of $\hat T_{\hat\chi}(\hat h)$, where

$$\hat h = \begin{pmatrix} \exp\{it\Delta^{1/2}/2\} & 0 \\ 0 & \exp\{-it\Delta^{1/2}/2\} \end{pmatrix}. \qquad (150)$$

In the same manner as for $\hat g$ of $\hat{QU}(2)$ we can represent

$$\hat h = \begin{pmatrix} e^{i\varphi\Delta^{1/2}/2} & 0 \\ 0 & e^{-i\varphi\Delta^{1/2}/2} \end{pmatrix}\Delta\begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] & -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] \\ -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] & g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \end{pmatrix} \qquad (151)$$

$$\times\Delta\begin{pmatrix} e^{\frac{i\psi\Delta^{1/2}}{2}} & 0 \\ 0 & e^{\frac{-i\psi\Delta^{1/2}}{2}} \end{pmatrix},$$

where φ, τ, and $\hat\psi$ are isoEuler angles of $\hat g_0$. From (153) we define $\hat T_{\hat\chi}(\hat g_\tau)$, namely,

$$\begin{pmatrix} g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] & -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] \\ -ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}] & g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \end{pmatrix}. \qquad (152)$$

Then, straighforward calculations yield [4]

$$\hat t^{\hat\chi}_{mn} = \frac{\Delta^{4+l}}{2\pi}\int_0^{2\pi} d\hat\theta\,(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}]-ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{i\theta\Delta^{1/2}\})^{l+n+\varepsilon}$$

$$\times(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \tag{153}$$

$$-ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{-i\theta\Delta^{1/2}\})^{l-n-\varepsilon}\exp\{i\theta(m-n)\Delta^{3/2}\}.$$

Introduce the function $B^l_{mn}(isocosh\hat{\tau})$ defining

$$B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) = \frac{\Delta^{4+l}}{2\pi}\int_0^{2\pi} d\hat{\theta}\,(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] - ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]$$

$$\times\exp\{i\theta\Delta^{1/2}\})^{l+n}(g_{11}^{-1/2}\cos[\frac{i\tau\Delta^{1/2}}{2}] \tag{154}$$

$$-ig_{22}^{-1/2}\sin[\frac{i\tau\Delta^{1/2}}{2}]\exp\{-i\theta\Delta^{1/2}\})^{l-n}\exp\{i\theta(m-n)\Delta^{3/2}\}.$$

Comparing (153) and (154) we have

$$\hat{t}^{\hat{\chi}}_{mn} = B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) \quad \hat{\chi} = (l,\varepsilon), \tag{155}$$

where

$$m' = m+\varepsilon, \quad n' = n+\varepsilon, \quad 0 \le \tau < \infty, \tag{156}$$

l is a complex number, m and n are simultaneously integer or half-integer numbers. From the expansion (153) it follows that

$$\hat{T}_{\hat{\chi}}(\hat{g}_0) = \hat{T}_{\hat{\chi}}(\hat{h}\varphi)\Delta\hat{T}_{\hat{\chi}}(\hat{g}_{\hat{\tau}})\Delta\hat{T}_{\hat{\chi}}(\hat{h}\psi). \tag{157}$$

So we can write

$$\hat{t}^{\chi}_{mn}(\hat{\varphi},\hat{\tau},\hat{\psi}) = \exp\{-i\Delta^{3/2}(m'\varphi + n'\psi)\}B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]), \tag{158}$$

where m',n', and τ' are defined according to (156). Since $B^l_{mn}(\hat{z})$ plays the same role for $\hat{QU}(2)$ as the function $P^l_{mn}(\hat{z})$ for $\hat{SU}(2)$, we call $B^l_{mn}(\hat{z})$ *isoJacobi function* of the variable $\hat{z} = g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]$.

11 IsoJacobi function $\hat{B}^l_{mn}(\hat{z})$

Integral representation of the isoJacobi function $B^l_{mn}(\hat{z})$ can be readily derived (see [4] for the usual case),

$$B^l_{mn}(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]) =$$

$$\frac{\Delta^{4+l}}{2\pi}\int_{-\tau}^{\tau}\frac{\exp\{(l-n+1/2)\Delta^{3/2}t\}}{\sqrt{2\Delta g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-\cos[it\Delta^{1/2}])}}\Big[\hat{z}_+^{m-n}\Delta(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}/2]$$

$$-\hat{z}_+ig_{22}^{-1/2}\sin[i\tau\Delta^{1/2}/2])^{2n}+\hat{z}_-^{m-n}\Delta(g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}/2] \tag{159}$$

$$-\hat{z}_-igg_{22}^{-1/2}\sin[i\tau\Delta^{1/2}/2])^{2n}\Big]d\hat{t},$$

where

$$\hat{z}_\pm=\frac{\exp\{\tau\Delta^{1/2}\}-g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}]}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}$$

$$\frac{\pm i\exp\{it\Delta^{1/2}/2\}\Delta\sqrt{2\Delta g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-\cos[it\Delta^{1/2}])}}{g_{22}^{-1/2}\sin[i\tau\Delta^{1/2}]}. \tag{160}$$

As one can see, the representation (159) is simplified when $n=m$ and also when $n=0$.

When $n=m$ we have directly from (159)

$$\hat{B}_{nn}^l(\hat{z})=\frac{\Delta^{5/2}}{\pi}\int_0^\tau\frac{g_{11}^{-1/2}\cos[(l-n+\frac{1}{2}\Delta^{1/2}]\Delta g_{11}^{-1/2}\cos[(2n\Delta^{3/2}\alpha)\Delta^{1/2}]d\hat{t}}{\sqrt{g_{11}^{-1}\cos[\frac{i\tau\Delta^{1/2}}{2}]+g_{22}^{-1}\sin[\frac{i\tau\Delta^{1/2}}{2}]}}. \tag{161}$$

When $n=0$ we have

$$\hat{B}_{m0}^l(\hat{z})=\frac{\Delta^{\frac{5}{2}+m}}{2\pi}\int_{-\tau}^{\tau}\frac{\exp\{(l+\frac{1}{2})\Delta^{3/2}t\}(\hat{z}_+^{\hat{m}}+\hat{z}_-^{\hat{m}})d\hat{t}}{2\Delta(g_{11}^{-1/2}(\cos[i\tau\Delta^{1/2}]-cos[it\Delta^{1/2}]))}. \tag{162}$$

Particularly, when in addition $m=0$ we have

$$\hat{B}_{00}^l(\hat{z})=\frac{\Delta^{3/2}}{\pi}\int_0^\tau\frac{\cos[i(l+1/2)t]dt}{\sqrt{\cos^{\hat{2}}[\frac{i\tau\Delta^{1/2}}{2}]-\cos^{\hat{2}}[\frac{it\Delta^{1/2}}{2}]}}. \tag{163}$$

12 IsoJacobi function $\hat{B}_{00}^l$

Let us put $m=n=0$ in (155). Then

$$\hat{t}_{00}^{\chi}(\hat{g}_0)=\hat{B}_{00}^l(\hat{z}). \tag{164}$$

We call $\hat{B}_{00}^l(\hat{z})$ isoJacobi function with index l and denote it simply $\hat{B}_l(\hat{z})$, namely,

$$\hat{B}_l(\hat{z})=\hat{t}_{00}^{\chi}(0,\hat{\tau},0,)=\hat{B}_{00}^l(\hat{z}), \tag{165}$$

where $\hat{\chi} = (l, 0)$ and $\hat{z} = g_{11}^{-1/2} \cos[i\tau\Delta^{1/2}]$.

The following integral representations for the isoJacobi function $\hat{B}^l(\hat{z})$ can be written:

$$\hat{B}_l(\hat{z}) = \frac{\Delta^{l+1}}{2\pi} \int_0^\pi (g_{11}^{-1/2} \cos[i\tau\Delta^{1/2}] - g_{22}^{-1/2} \sin[i\tau\Delta^{1/2}] g_{11}^{-1/2} \cos[i\theta\Delta^{1/2}])^l d\hat{\theta}, \tag{166}$$

$$\hat{B}_l(\hat{z}) = \frac{\Delta^{l+1}}{2\pi i} \int (g_{11}^{-1/2} \cos[i\tau\Delta^{1/2}] - i \frac{\hat{z}^2 + 1}{2\Delta^2 \hat{z}} g_{22}^{-1/2} \sin[i\tau\Delta^{1/2}])^l \frac{d\hat{z}}{\hat{z}}, \tag{167}$$

$$\hat{B}_l(\hat{z}) = \frac{\Delta^l \sqrt{2} g_{22}^{-1/2} \sin[\pi\Delta^{1/2} l]}{\pi} \int_0^\infty \frac{(g_{11}^{-1/2} \cos[i(l + \frac{1}{2})t\Delta^{1/2}] d\hat{t}}{\sqrt{\cos[it\Delta^{1/2}])^l + \cos[i\tau\Delta^{1/2}]}}. \tag{168}$$

From (166) it can be seen that when l is integer the isoJacobi function $\hat{B}_l(\hat{z})$ coincides with the isoLegendre polynomial,

$$\hat{B}_l(\hat{z}) = \hat{P}_l(\hat{z}), \tag{169}$$

which has been considered in Secs. 2-7.

12.1 Symmetry relations for $\hat{B}^l_{mn}(\hat{z})$ and $\hat{B}_l(\hat{z})$

Similarly to the isoLegendre polynomials $\hat{P}^l_{mn}(\hat{z})$, the isoJacobi functions $\hat{B}^l_{mn}(\hat{z})$ satisfy the following symmetry relations:

$$\hat{B}^l_{mn}(\hat{z}) = \hat{B}^l_{-m-n}(\hat{z}) \tag{170}$$

and

$$\hat{B}_l(\hat{z}) = \hat{B}_{-l-1}(\hat{z}). \tag{171}$$

13 Functional relations for $\hat{B}^l_{mn}(\hat{z})$

Functional relations for isoJacobi functions $\hat{B}^l_{mn}(\hat{z})$ can be derived in a similar fashion as it for isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$. Particularly, we have

$$\exp\{-i\Delta^{3/2}(m\varphi + n\psi)\} \hat{B}^l_{mn}(\hat{z}) = \sum_{k=-\infty}^{\infty} \exp -i\Delta^{3/2} k\varphi_2 \hat{B}^l_{mk}(\hat{z}_1) \Delta \hat{B}^l_{kn}(\hat{z}_2), \tag{172}$$

where $\hat{z} = g_{11}^{-1/2} \cos[i\tau\Delta^{1/2}]$, $\hat{z}_1 = g_{11}^{-1/2} \cos[i\tau_1\Delta^{1/2}]$, $\hat{z}_2 = g_{11}^{-1/2} \cos[i\tau_2\Delta^{1/2}]$, and τ, τ_1, τ_2, φ, and ψ are defined due to eqs. (136)-(138).

So, as a consequence of (172) we have the following particular cases.

(a) Let $\hat\varphi_2 = 0$, then $\hat\tau = \hat\tau_1 + \hat\tau_2$, $\hat\varphi = \hat\psi = 0$, and we have

$$\hat{B}^l_{mn}(g_{11}^{-1/2}\cos[i(\tau_1+\tau_2)\Delta^{1/2}]) = \sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta \tag{173}$$

$$\times \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]).$$

(b) Let $\hat\varphi_2 = \pi$, then $\hat\tau_1 \geq \hat\tau_2$, $\hat\tau = \hat\tau_1 - \hat\tau_2$, $\hat\varphi = 0$, $\psi = \pi$, and we have

$$\hat{B}^l_{mn}(g_{11}^{-1/2}\cos[i(\tau_1-\tau_2)\Delta^{1/2}]) = \sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta^2 \tag{174}$$

$$\times \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]).$$

(c) Particularly, when in addition $\hat\tau_1 = \hat\tau_2$, we have

$$\sum_{k=-\infty}^{\infty} \hat{B}^l_{mk}(g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}])\Delta^2 \hat{B}^l_{kn}(g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]) = \hat{B}^l_{mn}(1) \tag{175}$$

$$= \hat\delta_{mn} \equiv \delta_{mn}\Delta^{-1}.$$

Theorem of composition for isoLegendre function.

Let us define isoLegendre function and adjoint isoLegendre function as follows

$$\hat{B}_l(\hat z) = \hat{B}^l_{00}(\hat z) \tag{176}$$

and

$$\hat{B}^m_l(\hat z) = \frac{\hat\Gamma(l+m+1)}{\hat\Gamma(l+1)}\Delta \hat{B}^{m0}_l(\hat z), \quad \hat{B}^l_{m0}(\hat z) = \frac{\hat\Gamma(l+1)}{\hat\Gamma(l-m+1)}\Delta \hat{B}^{0m}_l(\hat z). \tag{177}$$

Putting $m = n = 0$ in (172) and using (176) and (177) we obtain

$$\hat{B}_l(\hat z) = \frac{\hat\Gamma(l-k+1)}{\hat\Gamma(l+k+1)}\Delta^3 e^{-i\Delta^{3/2}k\varphi_2}\hat{B}^k_l(\hat z_1)\hat{B}^k_l(\hat z_2), \tag{178}$$

where

$$g_{11}^{-1/2}\cos[i\tau\Delta^{1/2}] = g_{11}^{-1}\cos[i\tau_1\Delta^{1/2}]\Delta g_{11}^{-1}\cos[i\tau_2\Delta^{1/2}] + \tag{179}$$

$$g_{22}^{-1}\sin[i\tau_1\Delta^{1/2}]\Delta^2\sin[i\tau_2\Delta^{1/2}]g_{11}^{-1}\cos[i\varphi_2\Delta^{1/2}].$$

The composition formula for the adjoint isoLegendre function follows from (172) with $n = 0$, namely, we have

$$\hat{B}_l(\hat{z}) = \frac{\hat{\Gamma}(l+m+1)}{\hat{\Gamma}(l+k+1)} \Delta^3 e^{-i\Delta^{3/2}k\varphi_2} \hat{B}_l^k(\hat{z}_1)\hat{B}_l^k(\hat{z}_2), \tag{180}$$

where

$$\hat{\Gamma}(l+m+1) = \hat{\Gamma}(l+m)\Delta(l+m) and \hat{\Gamma}(l+m+1) = \int_0^\infty e^{-l-m}\Delta^2 \hat{x}^{l+m-1} dx. \tag{181}$$

Multiplication formula.

Multiplying both sides of the equation (172) by $\exp\{i\Delta^{3/2}k\varphi_2\}$ we obtain

$$\hat{B}^l_{mk}(\hat{z}_1)\hat{B}^l_{mk}(\hat{z}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} d\varphi_2\, e^{-i\Delta^{3/2}(k\varphi_2 - m\varphi - n\psi)} \hat{B}^l_{mn}(\hat{z}). \tag{182}$$

Putting $m = n = 0$ in (182) and using the symmetry relations we get

$$\hat{B}_l^k(\hat{z}_1)\hat{B}_l^{-k}(\hat{z}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} e^{-i\Delta^{3/2}k\varphi_2} \tag{183}$$

$$\hat{B}^l_{mn}(\hat{z}_1\Delta\hat{z}_2 + \hat{z}_3\Delta\hat{z}_4\Delta\hat{z}_5)d\varphi_2,$$

where $\hat{z}_1 = g_{11}^{-1/2}\cos[i\tau_1\Delta^{1/2}]$, $\hat{z}_2 = g_{11}^{-1/2}\cos[i\tau_2\Delta^{1/2}]$, $\hat{z}_3 = g_{11}^{-1/2}\sin[i\tau_1\Delta^{1/2}]$, $\hat{z}_4 = g_{11}^{-1/2}\sin[i\tau_2\Delta^{1/2}]$, and $\hat{z}_5 = g_{11}^{-1/2}\cos[i\varphi_2\Delta^{1/2}]$. Particularly,

$$\hat{B}_l(\hat{z}_1)\hat{B}_l(\hat{z}_2) = \frac{\Delta}{2\pi}\int_0^{2\pi} \hat{B}^l_{mn}(\hat{z}_1\Delta\hat{z}_2 + \hat{z}_3\Delta\hat{z}_4\Delta\hat{z}_5)d\varphi_2. \tag{184}$$

14 Recurrency relations for $\hat{B}^l_{mn}$

Recurrency relations for $\hat{B}^l_{mn}$ can be derived in the same manner as it for $\hat{P}^l_{mn}$. So, we do not represent the calculations here, and write down the final results.

$$\sqrt{\hat{z}^2-1}\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = \frac{(l+n)}{2}\Delta\hat{B}^l_{m,n-1}(\hat{z}) + \frac{(l-n)}{2}\Delta\hat{B}^l_{m,n+1}(\hat{z}), \tag{185}$$

$$\frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = -\frac{(l+n)}{2}\Delta\hat{B}^l_{m,n+l}(\hat{z}) + \frac{(l-n)}{2}\Delta\hat{B}^l_{m,n+1}(\hat{z}). \tag{186}$$

From (185) and (186) we have

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} - \frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = (l-n)\Delta\hat{B}^l_{m,n+1}(\hat{z}) \quad (187)$$

and

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} + \frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = (l+n)\Delta\hat{B}^l_{m,n+1}(\hat{z}). \quad (188)$$

Using the symmetry relations we have

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} + \frac{n-m\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = (l+m+1)\Delta\hat{B}^l_{m,n+1}(\hat{z}) \quad (189)$$

and

$$\sqrt{\hat{z}^2-1}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} + \frac{m-n\Delta\hat{z}}{\sqrt{\hat{z}^2-1}}\Delta\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} = (l-m+1)\Delta\hat{B}^l_{m,n+1}(\hat{z}). \quad (190)$$

Also,

$$(l-n)\hat{B}^l_{m,n+1}(\hat{z}) - (l+n)\hat{B}^l_{m,n-1}(\hat{z}) = \frac{2(m-n\Delta\hat{z})}{\sqrt{\hat{z}^2-1}}\Delta^2 B^l_{mn}(\hat{z}), \quad (191)$$

$$(l+m+1)\hat{B}^l_{m+1,n}(\hat{z}) - (l-m+1)\hat{B}^l_{m-1,n}(\hat{z}) = \frac{2(n-m\Delta\hat{z})}{\sqrt{\hat{z}^2-1}}\Delta^2 B^l_{mn}(\hat{z}). \quad (192)$$

The differential equation satisfied by isoJacobi function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}^l_{mn}(\hat{z})}{d\hat{z}^2} - 2z\frac{d\hat{B}^l_{mn}(\hat{z})}{d\hat{z}} - \frac{m^2+n^2-2mn\Delta}{z^2-1}\Delta\hat{B}^l_{mn}(\hat{z}) = l(l+1)\hat{B}^l_{mn}(\hat{z}). \quad (193)$$

The differential equation satisfied by adjoint isoLegendre function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}^m_l(\hat{z})}{d\hat{z}^2} - 2z\Delta\frac{d\hat{B}^m_l(\hat{z})}{d\hat{z}} - \frac{m^2\Delta^2}{z^2-1}\Delta\hat{B}^m_l(\hat{z}) = l(l+1)\Delta\hat{B}^m_l(\hat{z}), \quad (194)$$

and the equation satisfied by isoLegendre function is

$$\sqrt{\hat{z}^2-1}\frac{d^2\hat{B}_l(\hat{z})}{d\hat{z}^2} - 2z\Delta\frac{d\hat{B}_l(\hat{z})}{d\hat{z}} = l(l+1)\Delta\hat{B}_l(\hat{z}). \quad (195)$$

15 The group $\hat{M}(2)$

In this Section, we consider linear transformations of isoEuclidean plane.

15.1 Definitions

The motion of isoEuclidean plane $\hat{E}^2$ is similar to that of the ordinary Euclidean plane E^2 so the definition of the group $\hat{M}(2)$ is similat to that of $\hat{M}(2)$.

Choosing local coordinates $(\hat{x}, \hat{y})$ on $\hat{E}^2$, we write the motion $\hat{g} : (\hat{x}, \hat{y}) \rightarrow (\hat{x}', \hat{y}')$ in th following form:

$$\hat{x}' = \hat{x}\Delta g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}] - \hat{y}\Delta g_{22}^{-1/2}\sin[\alpha\Delta^{1/2}] + a, \qquad (196)$$

$$\hat{y}' = \hat{x}\Delta g_{22}^{-1/2}\sin[\alpha\Delta^{1/2}] - \hat{y}\Delta g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}] + b,$$

where

$$\hat{x} = g_{11}^{1/2}x, \quad \hat{y} = g_{22}^{1/2}y, \qquad (197)$$

so that, in an explicit form,

$$\hat{x}' = \hat{x}(g_{11}^{1/2}g_{22})\cos[\alpha\Delta^{1/2}] - \hat{y}(g_{11}^{1/2}g_{22})\sin[\alpha\Delta^{1/2}] + a, \qquad (198)$$

$$\hat{y}' = \hat{x}\Delta g_{11}^{3/2}\sin[\alpha\Delta^{1/2}] - \hat{y}\Delta(g_{11}^{1/2}g_{22})\cos[\alpha\Delta^{1/2}] + b.$$

Here, a, b, and α parametrize the motion $\hat{g}$ so that every element $\hat{g} \in \hat{M}(2)$ can be defined by the three parameters having the following ranges:

$$-\infty < a < \infty, \quad -\infty < b < \infty, \quad 0 \leq \alpha < 2\pi. \qquad (199)$$

Another realization of $\hat{M}(2)$ comes with the identification of $\hat{g}(a, b, \alpha)$ with the matrix

$$\hat{T}(\hat{g}) = \begin{pmatrix} g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}] & -g_{22}^{-1/2}\sin[\alpha\Delta^{1/2}] & a \\ g_{22}^{-1/2}\sin[\alpha\Delta^{1/2}] & g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}] & b \\ 0 & 0 & \Delta^{-1} \end{pmatrix}. \qquad (200)$$

It can be easily verified that

$$\hat{T}(\hat{g}_1)\Delta\hat{T}(\hat{g}_2) = \hat{T}(\hat{g}_1\Delta\hat{g}_1),$$

so that $\hat{T}(\hat{g})$ is a representation of $\hat{M}(2)$. This representation is an exact one, *i.e.* $\hat{T}(\hat{g}_1) \neq \hat{T}(\hat{g}_2)$ if $\hat{g}_1 \neq \hat{g}_2$. Thus, we conclude that the group $\hat{M}(2)$ is realized as group of 3×3 real matrices (201).

The group $\hat{M}(2)$ can be realized also as the group of 2×2 complex matrices. Namely, by the identification of $\hat{g}(a, b, \alpha)$ with the matrix

$$\hat{Q}(\hat{g}) = \left(\begin{array}{cc} \exp\{i\Delta^{3/2}\alpha\} & \hat{z} \\ 0 & \Delta^{-1} \end{array}) \right), \tag{201}$$

where

$$\hat{z} = a + i\Delta b. \tag{202}$$

It is easy to verify that $\hat{Q}(\hat{g}_1)\Delta\hat{Q}(\hat{g}_2) = \hat{Q}(\hat{g}_1\Delta\hat{g}_1)$ and $\hat{Q}(\hat{g}_1) \neq \hat{Q}(\hat{g}_2)$ if $\hat{g}_1 \neq \hat{g}_2$.

15.2 Parametrizations

For the parametrization above, let us find the composition law. Let $\hat{g}_1 = \hat{g}(a_1, b_1, \alpha_1)$ and $\hat{g}_2 = \hat{g}(a_2, b_2, \alpha_2)$. Then

$$\hat{T}(\hat{g}_1\Delta\hat{g}_2) = \tag{203}$$

$$\left(\begin{array}{ccc} g_{11}^{-1/2}\cos[\hat{\alpha}_1 + \hat{\alpha}_2] & -g_{22}^{-1/2}\sin[\hat{\alpha}_1 + \hat{\alpha}_2] & a_1 + a_2\Delta g_{11}^{-1/2}\cos[\hat{\alpha}_1] - b_2\Delta g_{22}^{-1/2}\sin[\hat{\alpha}_1] \\ g_{22}^{-1/2}\sin[\hat{\alpha}_1 + \hat{\alpha}_2] & g_{11}^{-1/2}\cos[\hat{\alpha}_1 + \hat{\alpha}_2] & b_1 + a_2\Delta g_{22}^{-1/2}\sin[\hat{\alpha}_1] + b_2\Delta g_{11}^{-1/2}\cos[\hat{\alpha}_1] \\ 0 & 0 & \Delta^{-1} \end{array} \right),$$

so that the law is

$$a = a_1 + a_2\Delta g_{11}^{-1/2}\cos[\alpha_1\Delta^{1/2}] - b_2\Delta g_{22}^{-1/2}\sin[\alpha_1\Delta^{1/2}], \tag{204}$$

$$b = b_1 + a_2\Delta g_{22}^{-1/2}\sin[\alpha_1\Delta^{1/2}] + b_2\Delta g_{11}^{-1/2}\cos[\alpha_1\Delta^{1/2}], \tag{205}$$

$$\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2. \tag{206}$$

Denoting $\hat{x} = (a_1, b_1)$ and $\hat{y} = (a_1, b_1)$ we rewrite the formulas (204)-(206) as follows:

$$\hat{g}(\hat{x}, \hat{\alpha})\Delta\hat{g}(\hat{y}, \hat{\beta}) = \hat{g}(\hat{x} + \hat{y}_\alpha, \hat{\alpha} + \hat{\beta}). \tag{207}$$

From this equation it follows that if $\hat{g} = \hat{g}(\hat{x}, \hat{\alpha})$ then

$$\hat{g}^{-1} = \hat{g}(-\hat{x}_{-\hat{\alpha}}, 2\pi - \hat{\alpha}). \tag{208}$$

Another useful parametrization can be represented by isoEuler angles. On the plane, we parametrize the vector $\hat{x} = (a, b)$ by isopolar angles $a = r\Delta g_{11}^{-1/2}\cos[\varphi\Delta^{1/2}]$ and $b = g_{22}^{-1/2}\sin[\varphi\Delta^{1/2}]$. The set of parameters for $\hat{g}$ is then $(\hat{r}, \hat{\varphi}, \hat{\alpha})$, with the rabges

$$0 \leq \hat{r} < \infty, quad 0 \leq \hat{\varphi} < 2\pi, quad 0 \leq \hat{\alpha} < 2\pi. \tag{209}$$

Decomposition for element of $\hat{M}(2)$ reads

$$\hat{g}(\hat{r},\hat{\varphi},\hat{\alpha}) = \hat{g}(0,\hat{\varphi},0)\Delta\hat{g}(\hat{r},0,0)\Delta\hat{g}(0,0,\hat{\alpha}-\hat{\varphi}). \tag{210}$$

Transformations corresponding to $\hat{g}(0,0,\hat{\varphi})$ and $\hat{g}(0,0,\hat{\alpha}-\hat{\varphi})$ are rotations while $\hat{g}(\hat{r},0,0)$ defines a parallel transport along the axis $O\hat{x}$. For $\hat{g}_1 = \hat{g}(\hat{r},0,\hat{\alpha}_1)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2,0,0)$, we have from eqs.(204)-(206)

$$\hat{g}_1\Delta\hat{g}_2 = \hat{g}(\hat{r},\hat{\varphi},\hat{\alpha})$$

, where

$$\hat{r} = \sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}} + 2\Delta\hat{r}_1\Delta\hat{r}_2\Delta g_{11}^{-1/2}\cos[\alpha\Delta^{1/2}]} \tag{211}$$

and

$$\hat{r}^{\hat{2}} = xb_1^2x + yb_2^2y + zb_3^2z; \hat{r}_1^{\hat{2}} = x_1b_1^2x_1 + y_1b_2^2y_1 + z_1b_3^2z_1, \tag{212}$$

$$\hat{r}_2^{\hat{2}} = x_2b_1^2x_2 + y_2b_2^2y_2 + z_2b_3^2z_2,$$

$$\exp i\Delta^{3/2}\varphi = \frac{\hat{r}_1 + \hat{r}_2\Delta\exp\{i\Delta^{3/2}\alpha_1}{\}}\hat{r}, \tag{213}$$

$$\hat{\alpha} = \hat{\alpha}_1. \tag{214}$$

To find the parameters of the composition $\hat{g}_1\Delta\hat{g}_2$ for $\hat{g}_1 = \hat{g}(\hat{r}_1,\hat{\varphi}_1,\hat{\alpha}_1)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2,\hat{\varphi}_2,\hat{\alpha}_2)$, one should replace $\hat{\alpha}_1$ by $\hat{\alpha}_1 + \hat{\varphi}_2 - \hat{\varphi}_1$, $\hat{\alpha}$ by $\hat{\varphi} - -\hat{\varphi}_1$, and $\hat{\alpha}$ by $\hat{\alpha} - \hat{\alpha}_2$ in (211)-(214).

From decomposition (210) and equation

$$\hat{g}(0,0,\hat{\alpha}_1 + \hat{\varphi}_2 - \hat{\varphi}_1) = \hat{g}(0,0,\hat{\alpha}_1 - \varphi_1)\Delta\hat{g}(0,0,\hat{\varphi}_2) \tag{215}$$

we get

$$\hat{g}_1\hat{g}_2 = \hat{g}(0,0,\hat{\varphi}_1)\hat{g}(\hat{r}_1,0,0,)\hat{g}(0,0,\hat{\alpha}_1+\hat{\varphi}_2-\hat{\varphi}_1)\hat{g}(\hat{r}_2,0,0,)\hat{g}(0,0,\hat{\alpha}_2+\hat{\varphi}_2)\Delta^3. \tag{216}$$

16 Irreps of $\hat{M}(2)$

16.1 Description of the irreps

Denote the space of smooth functions $f(\hat{x})$ on circle $x_1b_1^2x_1 + x_1b_1^2x_1 = \Delta^{-1}$ by D. To every element $\hat{g}(a,\hat{\alpha}) \in \hat{M}(2)$ we associate the operator $T_c(\hat{g})$ acting on $f(\hat{x})$,

$$\hat{T}_c(\hat{g})\hat{f}(\hat{x}) = e^{c\Delta(a,\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}}). \tag{217}$$

Here, c is fixed complex number, $\hat{x}_{-\hat{\alpha}}$ is vector to which the vector $\hat{x}$ is transformed by rotation on angle $-\hat{\alpha}$, and $(a, \hat{x}) = a_{11}x_1g_{11}^{1/2} + a_2x_2g_{11}^{1/2}$. Let us show that $T_c(\hat{g})$ is the representation of $\hat{M}(2)$. For $\hat{g}_1 = \hat{g}(a, \hat{\alpha})$ and $\hat{g}_2 = \hat{g}(b, \hat{\beta})$ we have

$$\hat{T}_c(\hat{g}_1)\hat{T}_c(\hat{g}_2)\hat{f}(\hat{x}) = \hat{T}_c(\hat{g}_1)e^{c\Delta(b,\hat{x})}\hat{f}(\hat{x}_{-\hat{\beta}})e^{c(a,\hat{x})}e^{c(b,\hat{x}_{-\hat{\alpha}}}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{218}$$

Since $(b, \hat{x}_{-\alpha}) = (\hat{b}_{\hat{\alpha}}, \hat{x})$ the following equation is valid:

$$\hat{T}_c(\hat{g}_1)\Delta\hat{T}_c(\hat{g}_2)\hat{f}(\hat{x}) = e^{c\Delta(a+b_{\hat{\alpha}},\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{219}$$

On the other hand, owing to (207)

$$\hat{g}_1\hat{g}_2 = \hat{g}(a, \hat{\alpha})\hat{g}(b, \hat{\beta}) = \hat{g}_(a + b_{\hat{\alpha}}, \hat{\alpha} + \hat{\beta}), \tag{220}$$

so that

$$\hat{T}_c(\hat{g}_1\Delta\hat{g}_2)\hat{f}(\hat{x}) = e^{c\Delta(a+b_{\hat{\alpha}},\hat{x})}\hat{f}(\hat{x}_{-\hat{\alpha}-\hat{\beta}}). \tag{221}$$

Thus, $\hat{T}_c(\hat{g}_1\Delta\hat{g}_2) = \hat{T}_c(\hat{g}_1\Delta\hat{T}_c\hat{g}_2)$, *i.e.* $\hat{T}_c(\hat{g})$ is representation of $\hat{M}(2)$.

Parametrical equations of the circle, $x_1b_1^2x_2 + x_2b_1^2x_2 = \Delta^{-1}$, have the form

$$x_1 = g_{11}^{-1}\cos[\psi\Delta^{1/2}], \quad x_2 = \Delta^{-1/2}\sin[\psi\Delta^{1/2}], \quad 0 \le \psi < 2\pi, \tag{222}$$

so that one can think of functions $f(\hat{x}) \in D$ as functions depending on $\hat{\psi}$,

$$\hat{f}(\hat{x}) = \hat{f}(\hat{\psi}). \tag{223}$$

The operator can be rewritten as

$$\hat{T}_c(\hat{g})\hat{f}(\hat{\psi}) = \exp\{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[(\psi - \varphi)\Delta^{1/2}]\}\hat{f}(\hat{\psi} - \hat{\alpha}), \tag{224}$$

where

$$a = (\hat{r}\Delta g_{11}^{-1/2}\cos[\varphi\Delta^{1/2}], \hat{r}\Delta g_{22}^{-1/2}\sin[\varphi\Delta^{1/2}]), \quad \hat{g} = \hat{g}(a, \hat{\alpha}).$$

By introducing scalar product,

$$(\hat{f}_1 \hat{;} \hat{f}_2) = \frac{\Delta^2}{2\pi}\int_0^{2\pi}\hat{f}_1(\hat{\psi})\bar{\hat{f}_2}(\hat{\psi})d\hat{\psi}, \tag{225}$$

we make the space D to be isoHilbert space $\mathcal{E}$. Then, $\hat{T}_c(\hat{g})$ is isounitary in respect to the scalar product (225) if and only if $c = \hat{i}\rho$ is an imaginary number.

16.2 Infinitesimal operators

The operator $\hat{T}_c(\hat{w}_1(\hat{t}))$, where

$$\hat{w}_1(\hat{t}) = \left(\begin{array}{ccc} \Delta^{-1} & 0 & t\Delta^{-1} \\ 0 & \Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-1} \end{array}) \right), \tag{226}$$

$w_1 \in \Omega_2$, transforms function $\hat{f}(\hat{\psi})$ to

$$\hat{T}_c(\hat{w}_1(\hat{t}))\hat{f}(\hat{\psi}) = \exp\{c\Delta^2 t\hat{r} g_{11}^{-1/2} \cos[\psi\Delta^{1/2}]\}\hat{f}(\hat{\psi}), \tag{227}$$

so that

$$\hat{A}_1 = \frac{d\hat{T}_c(\hat{w}_1(\hat{t}))}{d\hat{t}}|_{\hat{t}=0} = c\Delta g_{11}^{-1/2} \cos[\psi\Delta^{1/2}], \tag{228}$$

i.e. $\hat{A}_1$ acts as a multiplication operator.

Similarly, one can prove that the infinitesimal operator $\hat{A}_2$ corresponding to the subgroup Ω_2 represented by the matrices

$$\hat{w}_2(\hat{t}) = \left(\begin{array}{ccc} \Delta^{-1} & 0 & 0 \\ 0 & t\Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-1} \end{array}) \right) \tag{229}$$

is given by

$$\hat{A}_2 = C(g_{11}g_{22}^{1/2}) \sin[\psi\Delta^{1/2}]. \tag{230}$$

Also, for the subgroup Ω_3 consisting of the matrices

$$\hat{w}_3(\hat{t}) = \left(\begin{array}{ccc} g_{11}^{-1/2} \cos[t\Delta^{1/2}] & -g_{22}^{-1/2} \sin[t\Delta^{1/2}] & 0 \\ g_{22}^{-1/2} \sin[t\Delta^{1/2}] & g_{11}^{-1/2} \cos[t\Delta^{1/2}] & 0 \\ 0 & 0 & \Delta^{-1} \end{array}) \right) \tag{231}$$

we have

$$\hat{A}_3 = -\frac{d}{d\hat{\psi}}. \tag{232}$$

16.3 The irreps

The prove of irreducibility of the representation $\hat{T}(\hat{g})$ of the group $\hat{M}(2)$ can be given in the same way as it of $\hat{T}(\hat{g})$, and we do not present it here.

Below, we consider two choices of c.

(a) $c \neq 0$. We have

$$\hat{T}_c(\hat{w}_3(\hat{\alpha}))\hat{f}(\hat{\psi}) = \hat{f}(\hat{\psi} - \hat{\alpha}). \tag{233}$$

(b) $c = 0$. We have

$$\hat{T}_c(\hat{g})\Delta\hat{f}(\hat{\psi}) = \hat{f}(\hat{\psi} - \hat{\alpha}), \tag{234}$$

where $g = (\hat{x}\hat{,}\hat{\alpha})$. This representation is reducible since it can be decomposed into direct sum of the one-dimensional representations

$$\hat{T}_{0n}(\hat{g}) = e^{i\Delta^{3/2}n\alpha}. \tag{235}$$

Note that $\hat{T}_c(\hat{g})$ with $c \neq 0$ and $\hat{T}_{0n}(\hat{g})$, where n is integer number, constitute all possible irreps of $\hat{M}(2)$.

17 Matrix elements of the irreps of $\hat{M}(2)$ and isoBessel functions

17.1 Matrix elements

In the space $\mathcal{E}$, we choose the orthonormal basis $\{\exp(i\Delta^{5/2}n\psi)\}$ consisting of eigenfunctions of the operator $\hat{T}_c(\hat{w})$, $\hat{w} \in \Omega_3$. The matrix elements are written in this basis as

$$\hat{t}^c_{mn}(\hat{g}) = (\hat{T}_c(\hat{g})e^{in\psi\Delta^{3/2}}\hat{,}e^{im\psi\Delta^{3/2}}). \tag{236}$$

Taking into account definition (225) and eq.(224) we get

$$\hat{t}^c_{mn}(\hat{g}) = \frac{\exp\{-in\alpha\Delta^{3/2}\}}{2\pi}\Delta^3 \int_0^{2\pi} d\hat{\psi}\, e^{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[(\psi-\varphi)\Delta^{1/2}]}e^{i(n-m)\psi\Delta^{3/2}}. \tag{237}$$

Let $\hat{r} = \hat{\varphi} = 0$, *i.e.* $\hat{g}$ defines rotation on isoangle $\hat{\alpha}$. Due to orthogonality of the functions $\exp\{-in\psi\Delta^{5/2}\}$, we have

$$\hat{t}^c_{mn}(\hat{g}) \equiv \hat{t}^c_{mn}(\hat{\alpha}) = \exp\{-in\alpha\Delta^{3/2}\})\delta_{mn}. \tag{238}$$

Thus, the rotation is represented by a diagonal matrix $\hat{T}_c(\hat{\alpha})$, with non-zero elements being $\exp\{-in\alpha\Delta^{5/2}\}$, $\infty < n < \infty$.

Let $\hat{\varphi} = \hat{\alpha} = 0$. In this case, $\hat{g}$ defines transplacement on $\hat{r}$ along the $O\hat{x}$ axis so that (237) takes the form

$$\hat{t}^c_{mn}(\hat{g}) \equiv \hat{t}^c_{mn}(\hat{r}) = \frac{\Delta^2}{2\pi}\int_0^{2\pi} d\hat{\psi}\ \exp\{C\Delta^2\hat{r}g_{11}^{-1/2}\cos[\psi\Delta^{1/2}]+i(n-m)\psi\Delta^{3/2}\}. \tag{239}$$

Replacing $\hat{\psi}$ by $\pi/2 - \hat{\theta}$, we then have

$$\hat{t}^c_{mn}(\hat{r}) = \frac{\Delta^2}{2\pi} i^{n-m}\int_0^{2\pi} d\theta\ \exp\{c\Delta^2\hat{r}g_{22}^{-1/2}\sin[\theta\Delta^{1/2}] - i(n+m)\theta\Delta^{3/2}\}. \tag{240}$$

Let us denote

$$\hat{J}_n(\hat{x}) = \frac{\Delta^2}{2\pi} i^{n-m}\int_0^{2\pi} d\theta\ \exp\{\Delta^2 g_{22}^{-1/2}\sin[\theta\Delta^{1/2}] - in\theta\Delta^{3/2}\}, \tag{241}$$

and refer to $\hat{J}_n(\hat{x})$ as *isoBessel function*.

Using this definition we have from (240), in a compact writting,

$$\hat{t}^c_{mn}(\hat{r}) = i^{n-m}\Delta\hat{J}_{n-m}(-ic\Delta^2\hat{r}). \tag{242}$$

Now, to obtain $\hat{t}^c_{mn}(\hat{g})$ in a eneral case it is suffice to make the replacement $\hat{\psi} - \hat{\varphi} = \frac{\pi}{2} - \hat{\theta}$ in the integral (237). Namely, using (241) we obtain

$$\hat{t}^c_{mn}(\hat{g}) = i^{n-m}\Delta\exp\{-in\alpha\Delta^{3/2} + i(n+m)\varphi\Delta^{3/2}\}\hat{J}_{n-m}(-ic\Delta^2\hat{r}). \tag{243}$$

Indeed, from (243) it follows that

$$\hat{T}_c(\hat{g}) = \hat{T}_c(\hat{\varphi})\hat{T}_c(\hat{r})\hat{T}_c(\hat{\alpha} - \hat{\varphi}). \tag{244}$$

Since the matrices $\hat{T}_c(\hat{g})$ and $\hat{T}_c(\hat{\alpha} - \hat{\varphi})$ are both diagonal, with the non-zero elements $\exp\{i\Delta^{3/2}n\varphi\}$ and $\exp\{-i\Delta^{3/2}n(\varphi - \varphi)\}$ respectively, while $\hat{t}^c_{mn}(\hat{r}) = i^{n-m}\Delta\hat{J}_{n-m}(-ic\Delta^2\hat{r})$ we come to (243).

If $\hat{g}$ is an identity transformation, $\hat{g} = \hat{g}(0,0,0)$, then $\hat{T}_c(\hat{g})$ is the isounit matrix. Consequently, we have the following relations:

$$\hat{J}_{n-m}(0) = \hat{\delta}, \quad \hat{J}(0) = \Delta^{-1}, \quad \hat{J}_n(0) = 0, \qquad (n \neq 0).$$

17.2 IsoBessel functions with opposite sign indeces

In this section, we find the relation between the isoBessel functions with opposite sign indeces.

In the space of functions $\hat{f}(\hat{\psi})$, introduce the operator $\hat{Q}$ acting according to

$$\hat{Q}\Delta\hat{f}(\hat{\psi}) = \hat{f}(-\hat{\psi}). \tag{245}$$

This operator commutes with operator $\hat{T}_c(\hat{g}) \equiv \hat{T}_c(\hat{r})$, where $\hat{g} = \hat{g}(\hat{r}, 0, 0)$. Indeed,

$$\hat{T}_c(\hat{r})\Delta\hat{Q}\hat{f}(\hat{\psi}) = \hat{T}_c(\hat{r})\hat{f}(-\hat{\psi}) = \exp\{c\Delta^2\hat{r}g_{11}^{-1/2}\cos[\psi\Delta^{1/2}]\}\hat{f}(-\hat{\psi}).$$

Consequently,

$$\hat{Q}\hat{T}_c(\hat{r}) = \hat{T}_c(\hat{r})\hat{Q}. \tag{246}$$

Operator $\hat{Q}$ acts by changing the basis element, $\exp\{in\alpha\Delta^{5/2}\}$ to $\exp\{-in\alpha\Delta^{5/2}\}$, so the matrix has the form $(\hat{q}_{mn})$, where $\hat{q}_{m,-m} = \Delta^{-1}$ and $\hat{q}_{mn} = 0$ for $m + n \neq 0$. Thus, from (246) we obtain

$$\hat{t}^c_{-m,n}(\hat{r}) = \hat{t}^c_{m,-n}(\hat{r}). \tag{247}$$

Then, taking into account (242) we get

$$i^{m+n}\hat{J}_{n+m}(-i\Delta^2 c\hat{r}) = i^{-m-n}\hat{J}_{-n-m}(-i\Delta^2 c\hat{r}). \tag{248}$$

Putting in (248) $m = 0$ and $\hat{z} = -i\Delta^2 c\hat{r}$, we finally have

$$\hat{J}_n(\hat{z}) = -\Delta^{1-n}\hat{J}_{-n}(\hat{z}). \tag{249}$$

17.3 Expansion series for IsoBessel functions

Our aim is to derive the expansion series for isoBessel function in $\hat{x}$. To this end, we use integral representation (241). Expanding the exponent $\exp\{i\Delta^2\hat{x}g_{-1/2}\sin[\psi\Delta^2]\}$ and integrating over all the terms we obtain

$$\hat{J}_n(\hat{x}) = \sum_{k=0}^{\infty} a_k x^k (g_{11}^{k+1/2} g_{22}^k), \tag{250}$$

where

$$a^k = \frac{\Delta^{3k-s}}{2\pi k!}\int_0^{2\pi} d\psi\, \exp\{-i\Delta^{5/2}n\psi\}(i(g_{11}g_{22}^{1/2}\sin[\psi\Delta^2])^{\hat{k}}. \tag{251}$$

Here, $s = 1, 2, 3, \ldots$ On the other and, owing to the Euler formula,

$$(ig_{22}^{-1/2}\sin[\psi\Delta^2])^k = \frac{\exp\{i\Delta^{3/2}\psi}{-} e^{-i\Delta^{3/2}\psi}\}2^k\Delta^{2k} \tag{252}$$

$$= \sum_{m=0}^{k} \frac{(-1)^m \Delta^{2(1-k)-m} \hat{C}_k^m \exp\{i(k-2m)\Delta^{3/2}\psi\}}{2^k}.$$

Inserting this formula into (251), one can observe that a_k is non-zero iff $(k-n)$ is an even number, *i.e.* $k-n=2m$, $m \geq 0$. If $k=n+2m$ then

$$a_k = \frac{(-1)\Delta^m}{2^k m!(k-m)!\Delta^{k+2s}} = \frac{(-1)\Delta^{-n-3m-2s}}{2^{n+2m} m!(n+m)!}. \tag{253}$$

So, we finally have

$$\hat{J}_n(\hat{x}) = (g_{11}^{5/2-2s} g_{22}^{2-2s})(x/2)^n \sum_{m=0}^{k} \frac{-\Delta^{n-m} x^{2m}}{2^{2m} m!(n+m)!}. \tag{254}$$

18 Functional relations for isoBessel function

18.1 Theorem of composition

Theorem of composition for isoBessel function can be derived in the same manner as it for isoLegendre function $\hat{P}_{mn}^l$. One should use the equality $\hat{T}_c(\hat{g}_1 \Delta \hat{g}_2) = \hat{T}_c(\hat{g}_1)\Delta \hat{T}_c(\hat{g}_2$, that is

$$\hat{t}_{mn}^c(\hat{g}_1 \Delta \hat{g}_2) = \sum_{k=-\infty}^{\infty} \hat{t}_{mk}^c(\hat{g}_1)\Delta \hat{t}_{kn}^c(\hat{g}_2). \tag{255}$$

Let us put $\hat{g}_1 = \hat{g}(\hat{r}_1, 0, 0)$ and $\hat{g}_2 = \hat{g}(\hat{r}_2, \hat{\varphi}_2, 0)$. Then the parameters $\hat{r}$, $\hat{\varphi}$, and $\hat{\alpha}$ corresponding to the composition $\hat{g} = \hat{g}_1 \Delta \hat{g}_2$ can be expressed via parameters $\hat{r}_1$, $\hat{r}_2$, and $\hat{\varphi}_2$ as

$$\hat{r} = \sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}} + 2\hat{r}_1 \Delta \hat{r}_2 \Delta g_{11}^{-1/2} cos[\varphi_2 \Delta^{1/2}]}, \tag{256}$$

$$e^{i\Delta^{3/2}\varphi} = \hat{r}_1 + \hat{r}_2 \Delta e^{i\Delta^{3/2}\varphi_2}, \tag{257}$$

$$\alpha = 0, \tag{258}$$

where $\hat{r}_1^{\hat{2}}$, $\hat{r}_2^{\hat{2}}$, and $\hat{r}^{\hat{2}}$ are defined due to (212).

Inserting the matrix elements (243) into (255) and putting $m=0$ and $R = i\Delta^{-1}$ we have after some algebra

$$e^{i\Delta^{5/2}n\varphi} \hat{J}_n(\hat{r}) = \sum_{k=-\infty}^{\infty} e^{i\Delta^{5/2}k\varphi} \hat{J}_{n-k}(\hat{r}_1)\Delta \hat{J}_k(\hat{r}_2), \tag{259}$$

where $\hat{r}$, $\hat{r}_1$, $\hat{r}_2$, $\hat{\varphi}$, and $\hat{\varphi}_2$ are defined according to (256)-(257).

The formula (258) represents the *theorem of composition of isoBessel functions*.

Particularly, at $n = 0$ we have from (258)

$$\hat{J}_0(\hat{r}) = \sum_{k=-\infty}^{\infty} (\frac{-1}{\Delta^k}) e^{i\Delta^{5/2} k\varphi_2} \hat{J}_k(\hat{r}_1)\Delta^3 \hat{J}_k(\hat{r}_2). \tag{260}$$

Below, we consider some useful particular cases of the theorem.

(a) At $\hat{\varphi}_2 = 0$, we have $\hat{r} = \hat{r}_1 + \hat{r}_2$ and $\hat{\varphi} = 0$, so that

$$\hat{J}_n(\hat{r}_1 + \hat{r}_2) = \sum_{k=-\infty}^{\infty} \hat{J}_{n-k}(\hat{r}_1)\Delta \hat{J}_k(\hat{r}_2). \tag{261}$$

(b) At $\hat{\varphi}_2 = \pi$ and $\hat{r}_1 \geq \hat{r}_2$, we have $\hat{\varphi} = 0$ and $\hat{r} = \hat{r}_1 - \hat{r}_2$, so that

$$\hat{J}_n(\hat{r}_1 - \hat{r}_2) = \sum_{k=-\infty}^{\infty} (-1)\hat{J}_{n-k}(\hat{r}_1)\Delta^{1-k} \hat{J}_k(\hat{r}_2). \tag{262}$$

(c) For $\hat{\varphi} = \pi/2$, we have

$$(\frac{\hat{r}_1 + i\hat{r}_2}{\hat{r}_1 - i\hat{r}_2})^{\frac{n}{2}} \Delta^{\frac{n}{2}+1} \hat{J}_n(\sqrt{\hat{r}_1^{\hat{2}} + \hat{r}_2^{\hat{2}}}) = \sum_{k=-\infty}^{\infty} i^k \hat{J}_{n-k}(\hat{r}_1)\Delta^{1+k} \hat{J}_k(\hat{r}_2). \tag{263}$$

(d) For $\hat{r} = \hat{r}_1 = \hat{r}_2$ we have

$$\sum_{k=-\infty}^{\infty} \hat{J}_{n+k}(\hat{r})\Delta \hat{J}_k(\hat{r}) = \hat{J}_n(0) = \begin{matrix} \Delta^{-1}, & n = 0 \\ 0, & n \neq 0 \end{matrix} \tag{264}$$

18.2 Theorem of multiplication

Multiplying both sides of equation (259) by $\exp\{-i\Delta^{3/2} m\varphi_2\}/2\pi$ and integrating over $\hat{\varphi}_2$ in the range $(0, 2\pi)$, we have

$$\frac{\Delta^2}{2\pi} \int_0^{2\pi} e^{i\Delta(n\hat{\varphi} - m\hat{\varphi}_2} \hat{J}_n(\hat{r}) d\hat{\varphi}_2 = \hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2), \tag{265}$$

where $\hat{r}$, $\hat{r}_1$, $\hat{r}_2$, $\hat{\varphi}$, and $\hat{\varphi}_2$ are defined according to (256)-(258). Here, we have used the fact that $\exp\{i\Delta^{3/2} n\varphi_2\}$ are orthogonal so that all the terms are zero except for those with $k = m$.

The equation (265) represents the *theorem of product for isoBessel functions.*

Let us consider specific case of the theorem characterized by $\hat{r}_1 = \hat{r}_2 = R$. From this condition it follows that $\hat{r} = 2R\Delta^2 g g_{11}^{-1/2} cos[\frac{\varphi_2}{2}]$ and $\hat{\varphi} = \hat{\varphi}_2$, so that

$$\hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2) = \frac{\Delta^2}{2\pi}\int_0^{\pi} e^{i\Delta(n-2m)\hat{\varphi}}\hat{J}_n(2R\Delta^2 g_{11}^{-1/2}\cos[\frac{\varphi_2}{2}])d\hat{\varphi}. \quad (266)$$

Replacing the variable in the above integral by $\hat{r}$, we note that when $\hat{\varphi}_2$ varies from 0 to π the variable $\hat{r}$ varies from $\hat{r}_1 + \hat{r}_2$ to $|\hat{r}_1 - \hat{r}_2|$, while when $\hat{\varphi}_2$ varies from π to 2π the variable $\hat{r}$ varies from $|\hat{r}_1 - \hat{r}_2|$ to $\hat{r}_1 + \hat{r}_2$. In addition,

$$\frac{d\hat{r}}{d\hat{\varphi}_2} = \pm\frac{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)}}{2\Delta\hat{r}}, \quad (267)$$

where minus and plus signs correspond to $0 \leq \hat{\varphi}_2 \geq \pi$ and $\pi \leq \hat{\varphi}_2 \geq 2\pi$ respectively. Thus, we have

$$\hat{J}_{n-m}(\hat{r}_1)\hat{J}_m(\hat{r}_2) = \frac{2\Delta^2}{\pi}\frac{\int_{|\hat{r}_1-\hat{r}_2|}^{\hat{r}_1+\hat{r}_2} e^{i\Delta(n\hat{\varphi}-m\hat{\varphi}_2}\hat{J}_n(\hat{r})\hat{r}d\hat{r}}{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)^2}}, \quad (268)$$

where $\hat{\varphi}$ and $\hat{\varphi}_2$ are related to $\hat{r}$ according to (256)-(258).

At $m = n = 0$ the formula (268) takes the most simple form,

$$\hat{J}_0(\hat{r}_1)\hat{J}_0(\hat{r}_2) = \frac{2\Delta^2}{\pi}\frac{\int_{/\hat{r}_1-\hat{r}_2/}^{\hat{r}_1+\hat{r}_2}\hat{J}_0(\hat{r})\hat{r}d\hat{r}}{\sqrt{4\hat{r}_1^2\Delta\hat{r}_2^2 - (\hat{r}^2 - \hat{r}_1^2 - \hat{r}_2^2)}}. \quad (269)$$

19 Recurrency relations for $\hat{J}_n(\hat{z})$

As it for isoLegendre functions $\hat{P}^l_{mn}(\hat{z})$, recurrency relations for isoBessel functions follow from the composition theorem. Namely, we should first put $\hat{r}_2$ in this theorem to be infinitesimal.

Let us find derivatives of the isoBessel function on $\hat{x}$ at the point $\hat{x} = 0$. Differentiating (241) we have

$$\hat{J}'_n(0) = \frac{i\Delta^2}{2\pi}\int_0^{2\pi}\exp -i\Delta^2 n\hat{\psi} g_{22}^{-1/2}\sin[\hat{\psi}]d\hat{\psi} \quad (270)$$

$$= \frac{\Delta^2}{4\pi}\int_0^{2\pi}[\exp -i\Delta^2(n-1)\hat{\psi} - \exp -i\Delta^2(n+1)\hat{\psi}]d\hat{\psi}.$$

This integral is non-zero only when $n = \pm\Delta^{-1}$. Also,

$$\hat{J}'_1(0) = \hat{J}'_{-1}(0) = \frac{\Delta^{-1}}{2}. \tag{271}$$

Differentiating both sides of (263) on $\hat{r}_2$ and putting $\hat{r}_2 = 0$ we find

$$2\hat{J}'_n(\hat{x}) = \hat{J}_{n-1}(\hat{x}) - \hat{J}_{n+1}(\hat{x}). \tag{272}$$

Here, we used (271) and replace $\hat{r}$ by $\hat{x}$.

Similarly, from (265) we find

$$\frac{2n}{\hat{x}}\hat{J}_n(\hat{x}) = \hat{J}_{n-1}(\hat{x}) + \hat{J}_{n+1}(\hat{x}). \tag{273}$$

Combining (272) and (273) we finally obtain

$$\hat{J}_{n-1}(\hat{x}) = \frac{n}{\hat{x}}\Delta\hat{J}_n(\hat{x}) + \hat{J}'_n(\hat{x}), \tag{274}$$

$$\hat{J}_{n+1}(\hat{x}) = \frac{n}{\hat{x}}\Delta\hat{J}_n(\hat{x}) - \hat{J}'_n(\hat{x}) \tag{275}$$

. These formulas can be presented also in the following form:

$$\hat{J}_{n-1}(\hat{x}) = (\frac{n}{\hat{x}} + \frac{d}{d\hat{x}})\Delta\hat{J}_n(\hat{x}), \tag{276}$$

$$\hat{J}_{n+1}(\hat{x}) = (\frac{n}{\hat{x}} - \frac{d}{d\hat{x}})\Delta\hat{J}_n(\hat{x}). \tag{277}$$

20 Relations between IsoBessel functions and $\hat{P}^l_{mn}(\hat{z})$

20.1 IsoEuclidean plane and sphere

Two-dimensional sphere can be mapped to isoEuclidean plane in a standard way. Namely, this can be done in taking the limit $\hat{R} \to \infty$ for the radius of the sphere. Accordingly, $\hat{M}(2)$ can be considered as some limit of $\hat{SO}(3)$. More precisely, replacing $\hat{\varphi}$, $\hat{\psi}$, $\hat{\theta}$, $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\varphi}_2$ by $\hat{\varphi}$, $\hat{r}/\hat{R}$, $\hat{\alpha}$, $\hat{r}_1/\hat{R}$, $\hat{r}_2/\hat{R}$, and $\hat{\alpha}_1$ in (135) defining multiplications in $\hat{SO}(3)$ we should retain leading terms in the limit $\hat{R} \to \infty$. Simple calculations show that the result is exactly the formulas (211)-(214) defining multiplications in $\hat{M}(2)$.

20.2 IsoBessel and isoJacobi functions

The relation between the groups $\hat{M}(2)$ and $\hat{SO}(3)$ makes it possible to relate matrix elements of its irreducible isounitary representations. Thus, isoBessel functions, as matrix elements of representations $\hat{T}_{I\rho}(\hat{g})$ of $\hat{M}(2)$, can be derived from $\hat{P}^l_{mn}$, which are matrix elements of representations $\hat{T}_l(\hat{g})$ of $\hat{SO}(3)$. The limiting procedure is $\hat{R} \to \infty$ and $l \to \infty$.

To obtain concrete formulas we note first that $\hat{P}^l_{mn}$ has the integral representation,

$$\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\hat{\theta}]) = (\frac{\Delta^7}{2\pi})\sqrt{\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!}}\int_0^{2\pi} d\hat{\varphi}\,(g_{11}^{-1/2}\cos[\frac{\hat{\theta}}{2}]e^{i\hat{\varphi}/2}$$

$$+ig_{22}^{-1/2}\sin[\frac{\hat{\theta}}{2}]e^{-i\hat{\varphi}/2})(g_{22}^{-1/2}\sin[\frac{\hat{\theta}}{2}]e^{i\hat{\varphi}/2}+ig_{11}^{-1/2}\cos[\frac{\hat{\theta}}{2}]e^{-i\hat{\varphi}/2})e^{im\varphi}. \quad (278)$$

Putting $\hat{\theta} = \hat{r}/l$ and taking the limit $l \to \infty$ we find

$$\lim_{l\to\infty}\hat{P}^l_{mn}(g_{11}^{-1/2}\cos[\frac{\hat{r}}{l}]) = \frac{\Delta^{2(l+1)}}{2\pi}\int_0^{2\pi}(1+\frac{ir\Delta^{5/2}}{2l}\exp -i\varphi)^{l-n} \quad (279)$$

$$\times(1+\frac{ir\Delta^{5/2}}{2l}\exp i\varphi)\exp i\Delta^{3/2}(m-n)\varphi d(\varphi\Delta^{1/2}).$$

Note that at $m = n = 0$ the above relation takes the following simple form:

$$\lim_{l\to\infty}\hat{P}_l(g_{11}^{-1/2}\cos[\frac{\hat{r}}{l}\Delta^{1/2}]) = \hat{J}_0(\hat{r}), \quad (280)$$

so that $\hat{J}_0(\hat{r})$ appears as the limit from the isoLegendre polynomial.

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