NONPOTENTIAL TWO-BODY ELASTIC SCATTERING PROBLEM

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Abstract

In this paper, we consider elastic nonpotential scattering of proton off neutron, in triplet state S_1, within the framework of hadronic mechanics, which is used to account for effects not representable via the Hamiltonian. Among other results, we find that the angular distribution depends on the specific shape of the charge distribution of the nucleons.

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1 Introduction

In this paper, we consider elastic isotopic potential scattering of protons off neutrons, in triplet state S_1. This process is known to be interesting since electromagnetic interactions between p and n can be ignored. Up to the energy $10^4\ eV$, the section of the process does not practically change because only one partial wave with $l = 0$ should be accounted for. Indeed, at the energy $1\ MeV$ the wave length is $\lambda = 9.1 \times 10^{-13}\ cm$, that is more than the range of strong interactions, so that the assumption on the partial wave $l = 0$ is justified. It is beleived that such a situation holds up to the energy $10\ MeV$.

In the center-of-mass system, the scattering process $p+n \to p+n$ is equivalent to the scattering of one particle with the mass m in a static potential $V(r)$. We consider the scattering of neutrons off protons, with Yamaguchi potential.

In the framework of non-potential scattering theory[1, 2], we use the isotopically deformed Schrödinger equation, the iso-Schrödinger equation[3], characterized by the isotopic operator T. The operator T is assumed to be solely responsible for the *non-potential* part of the scattering. In the limit $T \to 1$, the usual potential scattering theory is recovered.

We study the formal solution of the iso-Schrödinger equation, the iso-Lippman-Schwinger equation, for the wave function $|\hat{\psi}^+\rangle$, from which the asymptotics of $|\hat{\psi}^+\rangle$, and the nonpotential scattering amplitude, $\hat{f}(\theta, \varphi)$, are obtained. Also, we calculate the free (iso-)propagator $\hat{G}_0^+$ in the coordinate and momentum representations. We solve the iso-Lippman-Schwinger equation for the scattering matrix, $\hat{J}(\hat{z}) = 4\pi \hat{f}(\theta, \varphi)$, rather than for $|\hat{\psi}^+\rangle$, to find the non-potential scattering amplitude $\hat{f}(\theta, \varphi)$ directly. The iso-scattering length $\hat{a}$ is defined from $\hat{a} = -\hat{f}(0)$. Experimental data for the process $p + n \to p + n$, are known to be (see for example [4])

$$a = -\frac{2}{\beta}\ \frac{(1+\kappa_0)^2}{1-(1+\kappa_0)^2},$$

where the parameters $\beta = 1.44\ Fm^{-1}$ and $\kappa_0 = 1/6.255$ remain mostly unexplained.

The paper is organized as follows.

In Section 2, we calculate the free propagator in the coordinate and momenta representations, to find the general solution of the iso-Schrödinger equation.

In Section 3, we consider the general solution of the iso-Schrödinger equation, indentify the non-potential scattering amplitude, and represent the solution and amplitude for particular choices of T.

In Section 4, we investigate the iso-Lippmann-Schwinger equation for the scattering matrix, its general on-shell solution for a separable potential, and non-potential scattering length.

In Section 5, we compute the iso-scattering length for the Yamaguchi potential, and indentify particular cases of the isotopic operator T.

In Section 6, we present a few concluding comments. We consider for simplicty isotopic elements T which are constant or depending at most on p.

2 The free propagator

We are working in the framework of the stationary (non-potential) scattering theory[1, 2, 3]. To find the asymptotics of the wave the function let us write the iso-Schrödinger equation in the form

$$(\hat{E}^{\pm} - H_0)T\hat{\psi} = \hat{V}\hat{\psi}, \tag{1}$$

where $\hat{V}(r)$ is a known function. Then, the general solution of the equation (1) reads

$$\hat{\psi}(\vec{r}) = \hat{\psi}_a(\vec{r}) + \int d\vec{r}'\hat{G}_0^+(\hat{E}^{\pm};\vec{r}-\vec{r}')T\hat{V}(\vec{r}')T\hat{\psi}(\vec{r}'), \tag{2}$$

where $\hat{\psi}_a$ is the general solution of the equation $(\hat{E}^{\pm} - H_0)T\hat{\psi}_a = 0$. Here, $\hat{G}_0^+(\hat{E}^{\pm};\vec{r}-\vec{r}')$ is the iso-Green function, which satisfies the following non-homogeneous equation:

$$(\hat{E}^{\pm} - H_0)T\hat{G}_0^+(\hat{E}^{\pm};\vec{r}-\vec{r}') = \hat{\delta}(\vec{r}-\vec{r}'), \tag{3}$$

where $\hat{E}^{\pm} = \lim_{\eta\downarrow 0}(E^{\pm} \pm \eta)\hat{I}$, $\hat{\delta}(\vec{r}-\vec{r}') = \hat{I}\delta(\vec{r}-\vec{r}')$, and $\hat{I} = T^{-1}$ is the isotopic unit operator.

From equation (3), straightforward calculations lead to the following form of the free (iso-)propagator $\hat{G}_0^+$:

$$\hat{G}_0^+ = \frac{\hat{I}}{\hat{z} - TH_0}, \tag{4}$$

and similarly one can find that

$$\hat{G}^{+} = \frac{\hat{I}}{\hat{z} - TH}, \tag{5}$$

where we have denoted $\hat{z} = \hat{E}^{+} - H_0$.

Now, we turn to representing coordinate and momentum representations of the free propagator. Throughout this paper, we assume that the isotopic operator T depends only on the momenta, $T = T(p)$.

Case 1: *r-representation.*

The matrix element is

$$\langle r|\hat{G}_0^{\pm}(\hat{z})|r'\rangle = \hat{I} * \langle r|\hat{G}_0^{\pm}(\hat{z})|r'\rangle = \sum_n \hat{I}\frac{\langle r|T|p\rangle T\langle p|T|r'\rangle}{\hat{z} - p^2T/2m} \to \langle r|\hat{G}_0^{\pm}(\hat{z})|r'\rangle, \tag{6}$$

where we have taken the limit $\sum_n \to \int$, and $\langle r|T|p\rangle = \exp[i\vec{p}T\vec{r}]$, $\langle p|T|r'\rangle = \exp[-i\vec{p}T\vec{r}']$. Then, the general form is

$$\langle r|\hat{G}_0^{\pm}|r'\rangle = 2m\int \frac{d^3p}{(2\pi)^3}\frac{\exp[i\vec{p}T(\vec{p})\vec{r}]\exp[-i\vec{p}T(\vec{p})\vec{r}']}{2m\hat{z} - p^2T(\vec{p})}. \tag{7}$$

Presenting $\Delta\vec{r} \equiv \vec{r} - \vec{r}'$ as $\Delta\vec{r} = -|p||\Delta r|\cos\theta$, and passing to moduls of vectors $\vec{r}$ and $\vec{p}$, we obtain from (7), integrating over θ, the final expression

$$\langle r|\hat{G}_0^{\pm}|r'\rangle = \frac{m}{2\pi^2}\frac{i}{\Delta r}\int_{-\infty}^{\infty}\frac{\exp[ipT(p)\Delta r]pdp}{(2m\hat{z} - p^2T(p))T(p)}. \tag{8}$$

We note that the integrand in (8) has different poles for different $T(p)$. So, further calculations in (8) require an explicit form of the dependence of the isotopic operator T on the momenta.

As an example, we calculate below the simplest possible case of $T(p) =$ const. The poles of the integrand in (8) are $p = \pm(2m\hat{z}/T)^{1/2}$, so that by standard technique,

$$\int_{-\infty}^{\infty}\frac{\exp[ipT\Delta r]pdp}{(2m\hat{z} - p^2T)T} = 2\pi i\ \mathrm{Res}\{\cdots\},$$

where

$$\mathrm{Res}\{\cdots\} = \frac{1}{2}\exp[ipT\Delta r]\hat{I}^{3/2},$$

and we obtain

$$\langle r|\hat{G}_0^{\pm}|r'\rangle = -\frac{m}{2\pi}T^{-3/2}\frac{\exp[ipT|r-r'|]}{|r-r'|}, \qquad (T = \text{const}). \tag{9}$$

Case 2: *p-representation.*
Taking into account that $|p\rangle T\langle p| = \hat{I}$, it is straightforward to find

$$\langle p|\hat{G}_0^{\pm}|p'\rangle = \frac{m}{2\pi^2}\frac{\delta(p-p')p^2T(p)}{2m\hat{z}-p^2T(p)}. \tag{10}$$

3 The general solution of the iso-Schrödinger equation

In a similar way as for the usual case (see for example Ref.[5]), having the general form of the r-representation (7) of the free Green function G_0^+, we can write the formal general solution of the iso-Schrödinger equation (1) as the iso-Lippman-Schwinger equation for $\hat{\psi}^+(\vec{r})$,

$$|\hat{\psi}^+(r)\rangle = |\hat{\psi}_a(r)\rangle + \int \frac{d^3r'}{2\pi^3}\langle r|\hat{G}_0^+(\hat{z})|r'\rangle T(p)\hat{V}(r')T(p)|\hat{\psi}^+(r')\rangle. \tag{11}$$

Using the r-representation of $\hat{G}_0^+(\hat{z})$ given by (8), we have from (11)

$$|\hat{\psi}^+(r)\rangle = |\hat{\psi}_a(r)\rangle + \int \frac{d^3r'}{2\pi^3}\frac{m}{2\pi^2}\frac{i}{\Delta r}\hat{V}(r')\int_{-\infty}^{\infty}\frac{\exp[ipT(p)\Delta r]pT(p)dp}{(2m\hat{z}-p^2T(p))}. \tag{12}$$

Again, we are led to perform further calculations for the particular choices of T. Below, we make calculations for the cases $T(p) = \text{const}$, and $T(p) = 1+\alpha^2p^2$, where α is a constant.

(a) The case $T(p) = \text{const}$.
With the use of (9) and (12), we obtain

$$|\hat{\psi}^+(r)\rangle = |\hat{\psi}_a^+(r)\rangle - \frac{m}{2\pi}\hat{I}^{1/2}\frac{\exp[ipTr]}{r}\int_0^{\infty}\frac{d^3r'}{(2\pi)^3}\exp[-ipTr']\hat{V}(r')|\hat{\psi}^+(r')\rangle, \tag{13}$$

where we have made the approximation $r \gg r'$; $|r - r'| = r(1 - \vec{r}\vec{r}/r^2) \approx r$. In the notation,

$$\langle r|\hat{p}^+\rangle = |\hat{\psi}^+(r)\rangle, \quad \langle r|\hat{p}\rangle = |\hat{\psi}_a(r)\rangle, \quad \langle r'|\hat{p}^+\rangle = |\hat{\psi}^+(r')\rangle,$$

and $n_r = \vec{r}/r^2$, the asymptotics (13) can be rewritten in a compact form,

$$\langle r|\hat{p}^+\rangle = \langle r|\hat{p}\rangle - \frac{m}{2\pi}\hat{I}^{1/2}\frac{\exp[ipTr]}{r}\langle n_r\hat{p}|r'\rangle\hat{V}(r')\langle r'|\hat{p}^+\rangle. \tag{14}$$

With the rearrangement $\langle n_r\hat{p}|r\rangle\hat{V}(r')\langle r'|\hat{p}^+\rangle = \langle n_r\hat{p}|V(r)|\hat{p}^+\rangle$, and the asymptotics (cf. Ref.[3])

$$\langle r|\hat{p}^+\rangle = \langle r|p\rangle + \hat{f}(\theta,\varphi)\frac{\exp[ipTr]}{r}, \tag{15}$$

it follows from (13) that

$$\hat{f}(\theta,\varphi) = -\frac{m}{2\pi}\hat{I}^{1/2}\int_0^\infty \frac{d^3r'}{(2\pi)^3}\exp[-ipTr']\hat{V}(r')\langle r'|\hat{p}^+\rangle, \tag{16}$$

where $\hat{f}(\theta,\varphi)$ is the *non-potential* scattering amplitude. Note that in the limit $T \to 1$, we recover the usual potential scattering amplitude $f(\theta)$.

(b) The case $T(p) = 1 + \alpha^2p^2$.
Substituting this $T(p)$ into (12), and performing the integral yields

$$|\hat{\psi}^+(r)\rangle = |\hat{\psi}_a^+(r)\rangle - \frac{m}{2\pi}\sqrt{1+\alpha^2p^2}\int_0^\infty \frac{d^3r'}{(2\pi)^3}\frac{\exp[ip\Delta r]}{\Delta r} \tag{17}$$

$$\times\exp[-ip^2\alpha^2\Delta r]\hat{V}(r')|\hat{\psi}^+(r')\rangle,$$

Then, similarly to the above case (a) we obtain for the non-potential scattering amplitude

$$\hat{f}(\theta,\varphi) = -\frac{m}{2\pi}\sqrt{1+\alpha^2p^2}\int_0^\infty \frac{d^3r'}{(2\pi)^3}\exp[-ip(1+\alpha^2p^2)r']\hat{V}(r')\langle r'|\hat{p}^+\rangle, \tag{18}$$

4 General solution to the iso-Lippmann-Schwinger equation. Non-potential scattering length

Let us start with iso-Lippmann-Schwinger equation for the (iso-)scattering matrix $\hat{J}(\hat{z})$,

$$\hat{J} = \hat{V} + \hat{J} * \hat{G}^{+} * \hat{V} \equiv \hat{V} + \hat{J}T\hat{G}^{+}T\hat{V}. \tag{19}$$

In the p-representation, this equation reads

$$\hat{I} * \langle p| * \hat{J} * \hat{I} * |q\rangle = \hat{I} * \langle p| * \hat{V} * \hat{I} * |q\rangle + \hat{I} * \langle p| * \hat{V} * \hat{I} * \hat{G}^{+} * \hat{J} * |q\rangle, \tag{20}$$

where p and q are initial and final momenta respectively. Taking into account that

$$\hat{I} * \langle p| * \hat{J} * \hat{I} * |q\rangle = \langle p|T\hat{J}T|q\rangle, \quad \hat{I} * \langle p| * \hat{V} * \hat{I} * |q\rangle = \langle p|T\hat{V}T|q\rangle, \tag{21}$$

$$\hat{I} * \langle p| * \hat{V} * \hat{I} * \hat{G}^{+} * \hat{J} * |q\rangle = \langle p|\hat{V}T\hat{G}^{+}T\hat{J}T|q\rangle = \langle p|T\hat{V}T|k\rangle T\langle k|T\hat{G}^{+}T\hat{J}T|q\rangle,$$

we have the following form for the iso-Lippmann-Schwinger equation:

$$\langle p|T\hat{J}T|q\rangle = \langle p|T\hat{V}T|q\rangle + \langle p|T\hat{V}T|k\rangle T\langle k|T\hat{G}^{+}T\hat{J}T|q\rangle \tag{22}$$

Let us denote $T\hat{J} = \hat{J}_0$, $T\hat{V} = \hat{V}_0$, and $T\hat{G}^{+} = \hat{G}_1^{+}$. Then the equation (22) can be rewritten

$$\langle p|\hat{J}_0T(p)|q\rangle = \langle p|\hat{V}_0T(p)|q\rangle + \langle p|\hat{V}_0T(p)|k\rangle T(k)\langle k|\hat{G}_1^{+}\hat{J}_0T(k)|q\rangle, \tag{23}$$

where we have explicitly indicated dependence of T on the momenta.

Using the p-representation (10), we obtain from (23) the following form of the equation for off-shell $\hat{J}(\hat{z})$:

$$\langle p|\hat{J}_0|q\rangle = \langle p|\hat{V}_0|q\rangle + \frac{m}{2\pi^2}\int_0^\infty \langle p|\hat{V}_0|k\rangle T(k)\langle k|\hat{J}_0T(k)|q\rangle \frac{T(k)k^2dk}{2m\hat{z} - k^2T(k)}. \tag{24}$$

For $\hat{z} = 0$, the equation (24) reduces to

$$\langle p|\hat{J}_0|q\rangle = \langle p|\hat{V}_0|q\rangle - \frac{m}{2\pi^2}\int_0^\infty \langle p|\hat{V}_0|k\rangle\langle k|\hat{J}_0T(k)|q\rangle dk. \tag{25}$$

We solve the equation (25) using separability of the potential, namely,

$$\langle p|\hat{J}_0|q\rangle \equiv \hat{J}_0^1(p)\hat{J}_0^2(q), \quad \langle p|\hat{V}_0|k\rangle \equiv \hat{\varphi}_0(p)\hat{\psi}(k), \tag{26}$$

$$\langle k|\hat{J}_0 T(k)|q\rangle \equiv \hat{J}_0^1(k)T(k)\hat{J}_0^2(q).$$

Substituting (26) into (25), and taking on-shell $\hat{J}(\hat{z})$, i.e. putting $|q\rangle = 0$, we have finally

$$\hat{J}(p) = \hat{\varphi}(p)(1 - \frac{m}{2\pi^2}A), \qquad (27)$$

where we have denoted

$$A = \int_0^\infty \hat{\psi}(k)\hat{J}_0^1(k)T(k)dk. \qquad (28)$$

As the last step, we perform the integral of both sides of the equation (27) with $T(p)\hat{\psi}(p)$, namely,

$$\int \hat{J}(p)T(p)\hat{\psi}(p)dp = \int \hat{\varphi}(p)T(p)\hat{\psi}(p)dp - \frac{m}{2\pi^2}A\int \hat{\varphi}(p)T(p)\hat{\psi}(p)dp. \qquad (29)$$

Denoting

$$B = \int_0^\infty \hat{\varphi}(p)T(p)\hat{\psi}(p)dp = \int_0^\infty \hat{V}(p,p)T(p)dp, \qquad (30)$$

we have $A = B - mA/2\pi^2$, that is

$$A = \frac{B}{1 + mB/\pi^2}. \qquad (31)$$

Thus, we have determined A entering the equation (27) in terms of the integral B defined by (30).

The scattering length, $\hat{a}$, can be then written as

$$\hat{a} = \hat{V}(0)\frac{1}{1 + mB/2\pi^2}, \qquad (32)$$

due to the relations $\hat{f}(\theta,\varphi) = \hat{J}(p)/4\pi$ and $\hat{a} = -\hat{f}(0)$.

5 Non-potential scattering length for specific $T(p)$

The integral B contains two characteristic entities, $V(p,p)$ and $T(p)$, particular forms of which should be specified to obtain final formulas for the iso-scattering length $\hat{a}$.

Below, we take Yamaguchi potential, $V(p,p)$, for the np-scattering[6],

$$V(p,p) = -\frac{8\pi}{\beta}\frac{(1+\kappa_0)^2}{(1+p^2)^2}, \tag{33}$$

and consider three particular cases of $T(p)$.

First, we specify the potential. The T-isotopic version of the Yamaguchi potential (33) is written as

$$\hat{V}(p,p) = -\frac{8\pi}{\beta} * \frac{(\hat{I}+\kappa_0)*(\hat{I}+\kappa_0)}{(\hat{I}+p*p)*(\hat{I}+p*p)} = -\frac{8\pi}{\beta}T\frac{(1+\kappa_0 T)^2}{(1+(pT)^2)^2}, \tag{34}$$

where we have merely made the T-isotopic liftings of the unit, $1 \to \hat{I}$, and products, $pp \to p*p$, etc. Note that when $T \to 1$ the potential (34) recovers the original potential (33). Then, the iso-scattering length (32) becomes

$$\hat{a} = \hat{V}(0)\frac{1}{1+B/4\pi^2}, \tag{35}$$

where

$$B = -\frac{8\pi}{\beta}W, \quad W \equiv \int_0^\infty \frac{(1+\kappa_0 T(p))^2 T(p)dp}{(1+p^2T^2(p))^2}, \tag{36}$$

and we have used $m = 1/2$.

Further calculations are performed with the Yamaguchi potential (34) for the following three choices of $T(p)$.

(a) The case $T(p) = T_0 = \text{const}$.

We have

$$\hat{a} = -\frac{2}{\beta}\,\frac{(1+\kappa_0)^2}{1-(1+\kappa_0 T_0)^2}, \tag{37}$$

(b) The case $T(p) = 1+\alpha^2p^2$ (α is a constant).

After tedious but straightforward calculations, we obtain

$$\hat{a} = -\frac{2}{\beta}\,\frac{(1+\kappa_0)^2}{1-((1+\kappa_0)^2(1+\alpha^2)-\alpha^2)}, \tag{38}$$

(c) The case $T(p) = 1+\cos(\alpha p)^n$ (α is a constant, $n \geq 2$). We have

$$\hat{a} = -\frac{2}{\beta}\,\frac{(1+\kappa_0)^2}{1-((1+\kappa_0)^2+8\kappa_0(1+\kappa_0)^2\sin(i\alpha)^n+8\kappa_0^2\sin 2(i\alpha)^n)}. \tag{39}$$

We note that in this case periodical $T(p)$ gives rise to complex valued scattering length.

6 Concluding remarks

As now established[3], the isotopic element T represents *internal* nonpotential effects which are averaged out into constants when conducting measures in the center-of-mass systems. The hadronic interpretion of the elastic scattering of protons and neutrons *in vacuum* in a triplet state can therefore only admit a constant value of T.

The issues to be addressed in this paper are therefore the identification of the difference between the potential and nonpotential scattering theories for the elastic scattering considered, and whether such difference can be resolved experimentally in favor of one or the other theory.

It is easy to see that *the nonpotential elastic scattering theory studied in this paper represents extended, generally nonspherical charge distributions of the proton and neutrons*. This results is achieved via the realization of isotopic element as the tensorial product $T = T_p \times T_n$, where T_p presents the shape of the charge distribution of the proton and T_n that of the neutron. By recalling that the particles are spinning, the shapes under consideration are oblate spheroidal ellipsoids. Under the assumption that the charge distributions of protons and neutrons are the same ($T_p = T_n$), the isotopic element T assumes the explicit form

$$T = \mathrm{diag}(b_1^2, b_2^2, b_3^2), \quad b_1^2 + b_2^2 + b_3^2 = 3,$$
$$b_1^2 = b_2^2 < b_3^2, \quad b_k = \mathrm{const} > 0, \tag{40}$$

where we have used the same normalization as occurring in the perfect sphere. Since deviations from the perfectly spherical shape are expected to be small, we then put

$$b_k^2 = 1 + \varepsilon_k, \quad \varepsilon_k \simeq 0. \tag{41}$$

The difference between the potential and nonpotential scattering theories is now evident. The conventional scattering amplitude and cross section are completely insensitive to the shape of the charge distributions of the protons and neutrons. On the contrary, the iso-scattering amplitude and isocross sections have an explicit dependence on the actual nonspherical shapes of the charge distributions of the scattering particles, resulting in a space anisotropy which depends on the amount of their oblateness. This result is due to the fact that all scalar products of the nonpotential scattering theory must be computed in the underlying iso-Euclidean space with iso-metric $\hat{\delta} = T\delta$, $\delta = \mathrm{diag}(1,1,1)$, implying the exponent of the iso-scattering amplitude (16),

$$pTr = p_1 b_1^2 r_1 + p_2 b_2^2 r_2 + p_3 b_3^2 r_3. \quad (42)$$

The related total iso-scattering cross section[3] has then a corresponding space anisotropy depending on oblateness of the spheroidal ellipsoids.

The above result is a mere operator version of the macroscopic scattering of two classical, identical, oblate, spheroidal and spinning objects with parallel intrinsic angular momenta.

We finally mention that the correct theoretical interpretation of the measured angles θ and φ requires the *isoangles* on iso-Euclidean space[3] which have been omitted in this paper for simplicity.

The experimental verifcation of the above prediction of nonpotential scattering theory requires a knowledge of the oblateness of the proton and of the neutron which is lacking at this writing in an experimental form (for prelminary theoretical predictions see[7]). Note that said oblateness could be derived from refined experimental measures on the predicted space anisotropy of the scattering distribution.

The case with an explicit dependence on momenta, $T = T(p)$, represents the scattering of protons and neutrons when *moving whithin a hadronics medium* (e.g., in the core of a star)[3]. As such, they are not usable for a study of the elastic scattering in vacuum, and will be considered for interior scattering problems at some future time.

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