

# FUNCTIONAL APPROACH TO PHASE SPACE FORMULATION OF QUANTUM MECHANICS<sup>1</sup>

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We present BRST gauge fixing approach to quantum mechanics in phase space. The theory is obtained by  $\hbar$ -deformation of the cohomological classical mechanics described by  $d = 1$ ,  $N = 2$  model. We use the extended phase space supplied by the path integral formulation with  $\hbar$ -deformed symplectic structure.

## 1 Introduction

Recently, in a series of papers Gozzi, Reuter and Thacker[1] developed path integral approach to classical mechanics. The physical states of the theory has been analyzed[1, 5, 6]. This theory can be viewed as one-dimensional cohomological field theory, in the sense that the resulting BRST exact Lagrangian is derived by fixing symplectic diffeomorphism invariance of the zero underlying Lagrangian. The BRST formulation of the cohomological classical mechanics has been given[7], and machinery of modern topological quantum field theories has been used[8, 9] to analyze the physical states, associated  $d = 1$ ,  $N = 2$  supersymmetric model, BRST invariant observables, and correlation functions. This field theoretic description provides a powerful tool to investigate the properties of Hamiltonian systems, such as ergodicity, Gibbs distribution, and Lyapunov exponents.

Further development can be made along the line of the phase space formulation of ordinary quantum mechanics originated by Weyl, Wigner and Moyal[12]. The key point one could exploit here is that it is treated[13]-[15] as a *smooth  $\hbar$ -deformation* of the classical mechanics. Indeed, there is an attractive possibility to give an explicit geometrical BRST formulation of the model describing quantum mechanics in phase space. The resulting theory

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could be thought of as a topological phase of quantum mechanics in phase space. The crucial part of the work has been done by Gozzi and Reuter[14] in the path integral formulation, where the associated extended phase space and quantum  $\hbar$ -deformed exterior differential calculus in quantum mechanics has been proposed. The core of this formulation is in the deforming of the Poisson bracket algebra of classical observables.

The central point we would like to use here is that the extended phase space can be naturally treated as the cotangent superbundle  $M^{4n|4n}$  over  $M^{2n}$  endowed with the second symplectic structure  $\Omega$  and (graded) Poisson brackets (4). Besides clarifying the meaning of the ISp(2) group algebra, appeared as a symmetry of the field theoretic model, it allows one, particularly, to combine symplectic geometry and techniques of fiber bundles. The underlying reason of our interest in elaborating the fiber bundle construction is that one can settle down Moyal's  $\hbar$ -deformation in a *consistent* way by using both of the Poisson brackets,  $\{, \}_\omega$  and  $\{, \}_\Omega$ . Namely, the two symplectic structures and Hamiltonian vector fields coexisting in the single fiber bundle are related to each other. Note that this relation is not direct since  $\{a^i, a^j\}_\omega = \omega^{ij}$  while for the projection of coordinates in the fundamental Poisson bracket  $\{\lambda^a, \lambda^b\}_\Omega = \Omega^{ab}$  to the base  $M^{2n}$  we have  $\{a^i, a^j\}_\Omega = 0$ . Also,  $Z_2$  symmetry of the undeformed Lagrangian can be used as a further important requirement for the deformed extension. Naively, the problem is to construct  $\hbar$ -deformed BRST exact Lagrangian, identify BRST invariant observables, and study BRST cohomology equation and corresponding correlation functions. Also, having the conclusion that the  $d = 1$ ,  $N = 2$  supersymmetry plays so remarkable role in the classical case it would be interesting to investigate its role in the quantum mechanical case.

In this way, one might formulate, particularly, quantum analogues of Lyapunov exponents in terms of correlation functions rather than to invoke to nearby trajectories, which make no sense in quantum mechanical case. The case of compact classical phase space corresponds to a finite number of quantum states. Also, we note that for chaotic systems expansion on the periodic orbits constitutes the only semiclassical quantization scheme known. Perhaps, this is a most interesting problem, in view of the recent studies of quantum chaos.

However, we should to emphasize here that the geometrical BRST analogy with the classical case is not straightforward, as it may seem at first glance, since one deals with non-commutative geometry[16] of the phase space in

quantum mechanical case (see Ref.[15] and references therein). Particularly, quantum mechanical observables of interest are supposed to be analogues of the closed  $p$ -forms on  $M^{2n}$ , with non-commuting coefficients arising to non-abelian cohomology.

In this paper, we present BRST gauge fixing approach to quantum mechanics in phase space.

The paper is organized as follows. In Sec. 2, we briefly present the BRST construction of the  $d = 1$ ,  $N = 2$  model characterizing classical dynamical systems. In Sec. 3, we introduce symplectic structure on the extended phase space naturally supplied by the field content. In Sec. 4, we make the  $\hbar$ -deformation of the symplectic structures using graded Moyal's brackets.

## 2 The BRST formulation

Starting point is the partition function[7]

$$Z = \int Da \exp iI_0, \quad (1)$$

where  $I_0 = \int dt \mathcal{L}_0$ , and the Lagrangian is trivial,  $\mathcal{L}_0 = 0$ . The basic field  $a^i(t)$  is the map from one-dimensional space  $M^1$  to  $2n$ -dimensional symplectic manifold  $M^{2n}$ . The BRST fixing of the symplectic diffeomorphism invariance yields[7]

$$Z = \int DX \exp iI, \quad (2)$$

where  $I = I_0 + \int dt \mathcal{L}_{gf}$ , and  $DX$  represents path integral over the fields entering the gauge fixing Lagrangian  $\mathcal{L}_{gf}$ . The Hamilton function  $\mathcal{H}$  associated to  $\mathcal{L}_{gf}$  has been found in the form

$$\mathcal{H} = q_i h^i + i\bar{c}_i \partial_k h^i c^k - \alpha a^i q_i - i\alpha c^i \bar{c}_i, \quad (3)$$

where  $c^i$  and  $\bar{c}_i$  are ghost and antighost fields, respectively,  $q_i$  is a Lagrange multiplier,  $h^i$  is Hamiltonian vector field, and  $\alpha$  is a real parameter. The Hamilton function (3) is BRST and anti-BRST exact. In the following, we use the delta function gauge by putting  $\alpha = 0$  in (3). We refer the reader to Refs. [7, 8, 9] for precise and detailed development of cohomological classical mechanics and geometrical meaning of the fields  $q_i$ ,  $c^i$  and  $\bar{c}_i$ .

### 3 Symplectic structures

In order to prepare for the  $\hbar$ -deformation, we need to identify geometrically the phase space and algebra of observables supplied by the BRST procedure.

Symplectic manifold  $M^{2n}$  can be viewed as cotangent fiber bundle  $(M^{2n}, M^n, T_x^*M^n, \omega)$  with the base space  $M^n$ , fiber  $T_q^*M^n$ , and fundamental symplectic two-form  $\omega$ , which in local coordinates,  $a^i = (p_1, \dots, p_n, x^1, \dots, x^n)$ ,  $a \in M^{2n}$ ,  $x \in M^n$ ,  $p \in T_x^*M^n$ , is  $\omega = \frac{1}{2}\omega_{ij}da^i \wedge da^j$ ;  $\omega_{ij} = -\omega_{ji}$ ,  $\omega_{ij}\omega^{jm} = \delta_m^i$ . In Hamiltonian mechanics,  $M^{2n}$  plays the role of phase space equipped by standard Poisson brackets,  $\{f, g\}_\omega = f\bar{\partial}_i\omega^{ij}\bar{\partial}_jg$ . We assume  $\omega_{ij}$  to be a constant matrix, i.e. use Darboux coordinates.

Taking the phase space  $M^{2n}$  as a base space, we consider a cotangent fiber bundle  $M^{4n}$  over it,  $(M^{4n}, M^{2n}, T_a^*M^{2n}, \Omega)$ , where two-form  $\Omega$  defines symplectic structure on  $M^{4n}$ , and is assumed to be closed,  $d\Omega = 0$ , and non-degenerate. In local coordinates on  $M^{4n}$ ,  $\lambda^a = (q_1, \dots, q_{2n}, a^1, \dots, a^{2n})$ ,  $\lambda \in M^{4n}$ ,  $a \in M^{2n}$ ,  $q \in T_a^*M^{2n}$ , the two-form  $\Omega$  is represented as  $\Omega = \frac{1}{2}\Omega_{ab}d\lambda^a \wedge d\lambda^b$ ;  $\Omega_{ab} = -\Omega_{ba}$ ,  $\Omega_{ab}\Omega^{bc} = \delta_c^a$ .

The cotangent bundle  $M^{4n}$  can be thought of as a second generation phase space equipped by the Poisson brackets,

$$\{F, G\}_\Omega = F\bar{\partial}_a\Omega^{ab}\bar{\partial}_bG, \quad (4)$$

where  $\partial_a = \partial/\partial\lambda^a$ , in view of the sequence

$$M^n \xrightarrow{p_0^{-1}} (M^{2n}, M^n, T_x^*M^n, \{\cdot, \cdot\}_\omega) \xrightarrow{p^{-1}} (M^{4n}, M^{2n}, T_a^*M^{2n}, \{\cdot, \cdot\}_\Omega) \quad (5)$$

Natural projection  $p$  is provided by  $p : (q, a) \mapsto (0, a)$ .

The ghost and antighost fields, as Grassmannian variables, can be naturally added to the symplectic structure on  $M^{4n}$  by enlarging  $M^{4n}$  to super-space  $M^{4n|4n}$ , with coordinates  $\tilde{\lambda}^k = (\lambda^a, \bar{c}_i, c^j)$ ,  $k, l = 1 \dots 8n$ , endowed with supersymplectic structure defined by the block diagonal matrix  $(\tilde{\Omega}^{kl}) = \text{diag}(\Omega^{ab}, I^{cd})$ , where  $I^{cd}$  is unit  $4n \times 4n$  matrix.

### 4 $\hbar$ -Deformed symplectic structures

Next step is to implement the  $\hbar$ -deformation. The  $\hbar$ -deformed version of the above symplectic structures is straightforward. Namely, we use the Moyal's

$\hbar$ -deformation[12] of the Poisson brackets which plays the role of algebra of quantum mechanical observables,

$$\{f, g\}_{\hbar\omega} = \frac{1}{i\hbar}(f * g - g * f) = f \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \bar{\partial}_i \omega^{ij} \bar{\partial}_j\right) g, \quad (6)$$

where the Moyal product is  $f * g = f \exp(\frac{i\hbar}{2} \bar{\partial}_i \omega^{ij} \bar{\partial}_j) g$ . In the classical limit, the  $\hbar$ -deformed product and brackets cover the usual pointwise product,  $f * g = fg + O(\hbar)$ , and the Poisson brackets,  $\{f, g\}_{\hbar\omega} = \{f, g\}_\omega + O(\hbar^2)$ , respectively.

The Lie-derivatives along the Hamiltonian vector field,  $\ell_h = h^i \partial_i$ , being linear maps, obey the conventional commutation relation,  $[\ell_{h_1}, \ell_{h_2}] = \ell_{[h_1, h_2]}$ , i.e. form a Lie algebra. The underlying algebra of Hamiltonian vector fields,  $[h_f, h_g] = h_{\{f, g\}_{\hbar\omega}}$ , is also a Lie algebra due to anticommutativity of the  $\hbar$ -deformed brackets (6). Quantum mechanical properties of the theory are thus encoded in these brackets, and the  $\hbar$ -deformation preserves the Lie algebra structure of the classical formalism, with the standard Lie-Poisson algebra replaced by the Lie-Moyal algebra.

Our aim is to exploit the symplectic structure  $\Omega$  on  $M^{4n}$  introduced above which appears to be crucial in finding the  $\hbar$ -deformed Hamilton function  $\mathcal{H}_\hbar$ .

Following the definition of the extended Moyal brackets[14], we use the symplectic structure on  $M^{4n}$  to define the  $\hbar$ -deformed Poisson brackets,

$$\{F, G\}_{\hbar\Omega} = F \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \bar{\partial}_a \Omega^{ab} \bar{\partial}_b\right) G, \quad (7)$$

with the underlying  $\hbar$ -deformed product  $F * G = F \exp(\frac{i\hbar}{2} \bar{\partial}_a \Omega^{ab} \bar{\partial}_b) G$ . Then, under the  $\hbar$ -deformation the sequence of maps (5) remains the same, with the Poisson brackets replaced by the  $\hbar$ -deformed Poisson brackets (6) and (7), respectively. Since the  $\hbar$ -deformed product of functions,  $f(a) * g(a)$  and  $F(\lambda) * G(\lambda)$ , is non-commutative, we deal in fact with *non-commutative* cotangent fiber bundles  $M^{2n}$  and  $M^{4n}$ , which can be studied in terms of non-commutative geometry[16].

Note that by rescaling  $q_i$  one has  $\{F, G\}_{\hbar\Omega} = \{F_\hbar, G_\hbar\}_{1\Omega}$ , where  $F_\hbar(a, q) \equiv F(a, \hbar q)$ , so that the deformation parameter  $\hbar$  can be assigned, equivalently, to functions on  $M^{4n}$  instead of the brackets. With the aid of the Grassmannian piece, the brackets (7) become the graded brackets

$$\{F, G\}_{\hbar\tilde{\Omega}} = F \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \bar{\partial}_k \tilde{\Omega}^{kl} \bar{\partial}_l\right) G, \quad (8)$$

where  $\partial_k = \partial/\partial\tilde{\lambda}^k$ . Here, the functions  $F, G, \dots$  are defined on the superspace  $M^{4n|4n}$ , and correspond to antisymmetric tensor fields and exterior forms on  $M^{2n}$ , which are (candidates to) observables of the theory.

The commutator of the Lie-derivatives on  $M^{2n}$ , together with the underlying  $\hbar$ -deformed algebra of Hamiltonian vector fields, can be represented as the  $\hbar$ -deformed Poisson brackets on  $M^{4n}$ ,  $[\ell_{h_1}, \ell_{h_2}] \leftrightarrow \{\ell_{h_1}, \ell_{h_2}\}_{\hbar\tilde{\Omega}}$ , for *horizontal* Hamiltonian vector fields on  $M^{4n}$ ,  $h^i(\lambda)$ , *i.e.* the fields orthogonal to the fibers  $T_a^*M^{2n}$ .

Due to coexistence of two symplectic structures,  $(M^{2n}, \omega)$  and  $(M^{4n}, \Omega)$ , the main point is to provide consistency between them. We require symplectic diffeomorphisms of the bundle  $(M^{4n}, \Omega)$  to preserve symplectic structure on the base  $(M^{2n}, \omega)$ . That is, under the natural projection, (i)  $p : \{, \}_{\hbar\tilde{\Omega}} \rightarrow \{, \}_{\hbar\omega}$  and (ii)  $p : h^a(\lambda) \rightarrow h^i(a)$ . Here, the Hamiltonian vector field on  $M^{4n}$  is  $h^a(\lambda) = \Omega^{ab}\partial_b H(\lambda)$  so that  $p : H(\lambda) \rightarrow H(a)$ .

The condition (i) in the form  $p : \{\mathcal{H}_{\hbar}, \rho(a)\}_{1\Omega} \rightarrow \{H(a), \rho(a)\}_{\hbar\omega}$  has been solved[14] to find the ghost-free part of  $\mathcal{H}_{\hbar}$ , where the projection  $p$  provides so called *horizontal condition*,  $q_i = \bar{c}_i = c^i = 0$ . We see that this condition naturally arises from the consistency requirements (i)-(ii). The ghost-dependent part of  $\mathcal{H}_{\hbar}$  is fixed uniquely due to the BRST invariance. Namely, the result is (cf.[14])

$$\mathcal{H}_{\hbar}(q, a) = q_i h_{\hbar}^i + i\bar{c}_i \partial_k h_{\hbar}^i c^k, \quad (9)$$

where

$$H_{\hbar}(q, a) = \frac{f(x)}{x} H(a) \equiv \int_{-1}^1 du \exp[-\hbar x u] H(a) \equiv \int_{-1}^1 du H(a^i - \hbar q_j \omega^{ij} u) \quad (10)$$

is  $\hbar$ -deformed classical Hamiltonian,  $f(x) = \text{sh}(x)$ ,  $x = \hbar q_i \omega^{ij} \partial_j$ , and  $h_{\hbar}^i = \omega^{ij} \partial_j H_{\hbar}(q, a)$  is  $\hbar$ -deformed Hamiltonian vector field.

The Hamilton function (9) is explicitly  $\hbar$ -deformed version of the Hamilton function (3), and plays the same role in the cohomological quantum mechanics as  $\mathcal{H}$  in cohomological classical mechanics[8]. It can be readily verified that in the ghost-free part it reproduces Wigner operator[15].

Generally, the states are defined by the  $p$ -ghost Wigner density  $\rho = \rho(a, c, t)$ , and the flow equation is  $\partial_t \rho = \{\rho, H\}_{\hbar\omega} = \mathcal{H}_{\hbar} \rho$ . In the classical limit,  $\hbar \rightarrow 0$ , the Hamilton function (9) reduces to the Hamilton function (3), and the flow equation reduces to the conventional Liouville equation, in the ghost-free part.

Due to the (anti-)BRST symmetry of the underlying classical theory (3), we should study the associated symmetry of the  $\hbar$ -deformed Hamilton function (9). The only difference from the classical case may arise from the BRST transformation of  $h_{\hbar}^i$ . Namely,  $\delta h_{\hbar}^i = \partial h_{\hbar}^i / \partial a^k \delta a^k + \partial h_{\hbar}^i / \partial q_k \delta q_k$ , and since  $sa^i = c^i$ ,  $sq_i = 0$ , we have  $sh_{\hbar}^i = \partial_k h_{\hbar}^i c^k$ . This means that the BRST symmetry survives the  $\hbar$ -deformation, and  $calH_{\hbar}$  is BRST invariant,  $s\mathcal{H}_{\hbar} = 0$ .

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