SUPERSYMMETRIC HADRONIC MECHANICAL HARMONIC OSCILLATOR

A.K.Aringazin
Department of Theoretical Physics
Karaganda State University
Karaganda 470074, USSR

1990

Abstract

Lie-isotopic lifting of the commutation and anticommutation relations of creation-annihilation bosonic and fermionic operators is considered. Lie-isotopic generalization of the associated supersymmetry generators and of the underlying superalgebra in considering of the Lie-isotopically lifted hamiltonian of supersymmetric quantum mechanical oscillator is defined. The generalised oscillator has the properties similar to that of the conventional supersymmetric oscillator: the ground state is characterised by null energy while the other energy levels are twice degenerated. A possible Lie-isotopic generalization of supersymmetric quantum mechanics is briefly discussed.

Hadronic J. **13** (1990) 263-276.

1 Introduction

In a recent series of papers Mignani, Myung and Santilli[1, 2, 3, 4, 5] developed Lie-isotopic generalization of the conventional quantum mechanics. The generalization called hadronic mechanics is based on the Lie-isotopic lifting of the operator theory consisting of: the liftings of an associated complex algebra of operators, $\mathcal{E}$, field of complex numbers, C, and of quantum mechanical Hilbert space, $\mathcal{H}$. In the Lie-isotopic formulation, the lifting $\mathcal{E} \to \hat{\mathcal{E}}$ is characterised by introducing of the product $A * B \equiv ATB$, where T is hermitian, invertible and fixed element of $\mathcal{E}$, and of the new unit $I^* = T^{-1}$. The antisymmetric algebra attached to the isotope $\hat{\mathcal{E}}$ is a Lie-isotopic algebra with the generalized bracket $[A, B]^* = A * B - B * A$.

Myung and Santilli[1] presented a generalization of all familiar operations on a conventional Hilbert space, with the compatible generalized fundamental equations being iso-Heisenberg equation and iso-Schrodinger equation[4, 5].

Various aspects of the hadronic mechanics have been studied by Nishioka[6]. For a brief survey of the literature on the Lie-isotopic theory see Ref. [7].

Explicit calculations[8] showed that under a natural Lie-isotopic lifting of the anticommutation algebra of creation-annihilation operators and of the number operator, the Pauli principle is garanteed for a composite system in the exterior branch of hadronic mechanics, in a way compatible with possible departure from the principle for each spin-half constituents. A similar result for the Heisenberg uncertainty principle had been established earlier by Mignani, Myung and Santilli[5]

In this paper, we consider Lie-isotopic liftings of the commutation and anticommutation of the creation-annihilation bosonic anf fermionic operators, respectively, in order to study a Lie-isotopic generalized of supersymmetric quantum mechanical harmonic oscillator. We shall show that the associated hamiltonian, supersymmetric generator, and superalgebra can be naturally generalized in a Lie-isotopic way, with the ground state energy of the generalized oscillator being null while the other energy levels being twice degenerate.

In section 2 we briefly recall the basic definitions and properties of bosonic, fermionic and supersymmetric oscillators. Lie-isotopic liftings of these oscillators are defined in section 3. We call the lifted oscillators as hadronic mechanical ones following the treatment that the Lie-isotopic liftings of the conventional quantum mechanics (QM) is refered to as hadronic mechanics (HM). WE show explicitly that the HM bosonic oscillator has the same energy spectrum as the conventional QM one. The HM fermionic oscillator has been already found[8] to have the same energy spectrum as the QM one (Pauli principle is garanteed). Supersymmetry properties for the composite system consisting of HM bosonic and HM fermionic oscillators take place when one uses the same element T for the lifting of both the bosonic and fermionic oscillators, and define then the corresponding lifting of supersymmetry generators via this T.

In section 4 we discuss a possible Lie-isotopic generalization of supersymmetric quantum mechanics following the convwntional procedure of including "boson-boson" and "boson-fermion" interactions to the "free" theory. In section 5 we briefly discuss some open problems of the supersymmetric hadronic mechanics.

2 Supersymmetric harmonic oscillator

In the section we briefly recall one of the main problems of supersymmetric quantum mechanics [9, 10] (SUSY QM), namely, the problem of supersymmetric harmonic oscillator.

(a) QM bosonic oscillator.

Hamiltonian of the usual QM harmonic bosonic oscillator,

$$H_B = \frac{1}{2}p^2 + \frac{1}{2}\omega_B^2 q^2, \tag{1}$$

where the momentum and the coordinate operators, p and q, satisfy the canonical commutation realtions, $[p, q] = 1$, can be presented in terms of bosonic creation-annihilation operators, b^+ and b, as follows:

$$H_B = \frac{1}{2}\omega_B\{b^+, b\}, \tag{2}$$

where

$$b = \frac{1}{\sqrt{2\omega_B}}(ip + \omega_B q), \tag{3}$$

and the following commutation relations hold:

$$[b, b^+] = 1, \quad [b, b] = 0, \quad [b^+, b^+] = 0. \tag{4}$$

Operatores b and b^+ act on the state $|n_B>$, and the energy spectrum of the hamiltonian (2) is well known to be

$$E_B = \omega_B(n_B + \frac{1}{2}), \quad n_B = 0, 1, 2, ... \tag{5}$$

where n_B are the eigenstates of the bosonic number operator $N_B = b^+b$.

(b) QM fermionic oscillator.

Fermionic creation-annihilation operators, f^+ and f, obey the anticommutation relations,

$$\{f, f^+\} = 1, \quad \{f, f\} = 0, \quad \{f^+, f^+\} = 0. \tag{6}$$

Hamiltonian of the fermionic oscillator can be constructed, by the analogy with the bosonic hamiltonian (2), as

$$H_F = \frac{1}{2}\omega_F[b^+, b]. \tag{7}$$

Operators f and f^+ act on the state $|n_F>$, and the energy spectrum of the fermionic oscillator can be calculated by using of the anticommutation relations (6), namely,

$$E_F = \omega_F(n_F - \frac{1}{2}), \quad n_F = 0, 1 \tag{8}$$

where $n_F = 0, 1$ (Pauli principle) are the eigenvalues of the fermionic number operator $N_F = f^+f$.

(c) QM supersymmetric oscillator.

For the composite system of the bosonic and fermionic oscillators, $\omega = \omega_B = \omega_F$, we get from (5) and (8)

$$E = \omega(n_B + n_F), \tag{9}$$

This formula implies that all energy levels of the system are twice degenerate except for the ground state, $n_B = n_F = 0$, characterized by zero energy, $E = 0$. Generators of the symmetry ($n_B \to n_B \mp 1, n_F \to n_F \pm 1$) responsible for the degeneracy are supersymmetry generators,

$$Q_- = \sqrt{2\omega}b^+f, \quad Q_+ = \sqrt{2\omega}bf^+. \tag{10}$$

They transform "bosons" to "fermions", and vice versa. $Q_+|n_B, n_F>\sim |n_B-1, n_F+1>$, $Q_-|n_B, n_F>\sim |n_B+1, n_F-1>$, and are nilpotent, $Q_+^2 = Q_-^2 = 0$. Supersymmetry generators (10) commute with the hamiltonian of the supersymmetric oscillator,

$$H = H_B + H_F \equiv \omega(b^+b + f^+f) \tag{11}$$

namely,

$$[Q_\pm, H] = 0. \tag{12}$$

Thus, the hamiltonian (11) is supersymmetric. Further, anticommutator between Q_- and Q_+ is proprtional to hamiltonian,

$$\{Q_-, Q_+\} = 2H. \tag{13}$$

In terms of anticommuting hermitian operators $Q_{1,2}$,

$$Q_1 = Q_+ + Q_-, \quad Q_2 = -i(Q_+ - Q_-) \tag{14}$$

the hamiltonian (11) can be rewritten as $h = Q_1^2 = Q_2^2$, and, again, operators $Q_{1,2}$ commute with H,

$$[Q_i, H] = 0. \tag{15}$$

i.e. they are supersymmetry generators, and

$$\{Q_i, Q_k\} = 2\delta_{ik}H, \quad i,k = 1,2. \tag{16}$$

The last two relations are known as presenting a simplest Lie's superalgebra.

Aupersymmetry is a dynamical symmetry since one of the generators of it is a hamiltonian. Note also that the SUSY hamiltonian H should have a non-negative energy spectrum since it can be presented as a square of the hermitian operators Q_1 or Q_2.

3 Lie-isotopic lifting of the SUSY QM oscillator

In this section we define Lie-isotopic liftings of the hamiltonian (11) of the SUSY oscillator and the SUSY generators (10). We shall show that the generalized hamiltonian and SUSY generators verify the Lie-isotopically lited superalgebra relations, similar to the basic relations (12) and (13), if the underlying commutation and anticommutation relations for creation-annihilation bosonic and fermionic operators have been lifted by the same element T. It is worthwhile to note that the latter requirement is similar to that of coincideness of the frequinces ω_B and ω_F, to gain supersymmetry.

(a) HM bosonic oscillator.

Define the Lie-isotopic lifting of the commutation relations (4) as follows:

$$[b, b^+]^* \equiv b * b^+ - b^+ * b = 1, \quad [b, b]^* = 0, \quad [b^+, b^+]^* = 0 \tag{17}$$

where $b * b^+ \equiv bTb$ and $I^* = T^{-1}$, Here, we have merely replaced the commutator and the unit by the Lie-isotopic commutator and the iso-unit, respectively. In the limit $T \to 1$ the Lie-isotopic commutation relations (17) recover the conventional ones (14).

The lifting of the hamiltonian (2) of the bosonic oscillator can be naturally defined as

$$H_B = \frac{1}{2}\omega_B\{b^+, b\}^*, \tag{18}$$

where $\{b, b^+\}^* \equiv b * b^+ + b^+ * b = 1$. With the following lifting of the bosonic number operator:

$$N_B = b^+ * b \tag{19}$$

which is evidently iso-hermitian, $N_B^+ = N_B$, nd the Lie-isotopic commutation relations (17), the hamiltonian (18) can be rewritten as

$$H_B = \omega_B(N_B + \frac{1}{2}). \tag{20}$$

Let us consider the iso-eigenvalue problem[3] for the Lie-isotopic bosonic number operator (19),

$$N_B * |n_B >= \hat{n}_B * |n_B >= n_B|n_B > . \tag{21}$$

Here, the iso-modular action of the operator N_B on iso-ket vector $*| >$ of the iso-Hilbert space is presented, and $\hat{n}_B$ is an iso-number, $\hat{n}_B \equiv n_B I^*$, $n_B \in C$.

Let us examine now are the eigenvalues n_B of N_B integer, as it is for the case of the convational bosonic number opearotr N_B of section 2. Using the commutation relations (17), we find, following the conventional procedure, that

$$[N_B, b^+]^* = b^+ \tag{22}$$

so that

$$b^+ * N_B = (N_B - I^*) * b^+$$

$$b^+ * b^+ * b * b = b^+ * N_B * b = N_B * (N_B - I^*), \quad \text{etc.}$$

and then, in general, for k operators,

$$< | * b^+ * ... * b^+ * b * ... * b * | >= n_B(n_B - 1)...(n_B - k + 1) < | * | > . \quad (23)$$

Thus, since the iso-norm $< | * | >$ is not negative, and the eigenvalues of the iso-hermitian opearotr are real[3], we arrive at the conclusion that $n_B = 0, 1, 2, ...$ exactly as in the usual Bose case (see Eq. (5)). Therefore, the energy spectrum for the HM bosonic oscillator has the form

$$E_B = \omega_B(n_B + \frac{1}{2}), \quad n_B = 0, 1, 2, ... \quad (24)$$

(b)HM fermionic oscillator.

In Ref.[8] the following Lie-isotopic lifting of the anticommutation relations of the creation-annihilation fermionic operators, f^+ and f, has been defined:

$$\{f, f^+\}^* = I^*, \quad \{f, f\}^* = 0, \quad \{f^+, f^+\}^* = 0. \quad (25)$$

The eigenvalues of the Lie-isotopically lifted fermionic number operator, $N_F = f^+ * f$, are $n_F = 0, 1$ (validity of the Pauli principle in the exterior branch of hadronic mechanics). This is shown explicitly in Ref. [8] and will not be repeated here.

A natural Lie-isotopic lifting of the conventional hamiltonian (7) of the QM fermionic oscillator can be written as

$$H_F = \frac{1}{2}\omega_F[b^+, b]^* \quad (26)$$

so that the energy spectrum turns out to be of the same form as in the usual Fermi case,

$$E_F = \omega_F(n_F - \frac{1}{2}), \quad n_F = 0, 1. \quad (27)$$

(c) HM supersymmetric oscillator.

As in section 2 let us put $\omega = \omega_B = \omega_F$, and study a composite HM bosonic-fermionic oscillator with the hamiltonian

$$\hat{H} = H_B + H_F \equiv \omega(b^+ * b + f^+ * f). \quad (28)$$

According to Eqs.(24) and (27) the energy spectrum of the composite system appears to be

$$E = \omega(n_B + n_F), \tag{29}$$

so that, again, the twice degeneracy of all the energy levels takes pleace except for the ground state, $n_B = n_F = 0$, characterized by zero energy, $E = 0$.

By the analogy with the liftings of the number operators made in this section, one can define the following Lie-isotopic liftings of the SUSY generators (10):

$$\hat{Q}_- = \sqrt{2\omega}b^+ * f, \quad \hat{Q}_+ = \sqrt{2\omega}b * f^+. \tag{30}$$

Using the commutation relations $[b^+, f]^* = [b, f^+]^* = 0$, one can easily verify that the newoperators $\hat{Q}_\pm$ are iso-nilpotents, $\hat{Q}_+ * \hat{Q}_+ = \hat{Q}_- * \hat{Q}_- = 0$, and are conjugated to each other, $\hat{Q}_+^+ = \hat{Q}_-$, $\hat{Q}_-^+ = \hat{Q}_+$. These quantities are indeed generators of a symmetry since they commute with the hamiltonian $\hat{H}$,

$$[\hat{Q}_\pm, \hat{H}] = 0 \tag{31}$$

where the use of the relation $[f, b^+]^* = 0$ has been made. Further, they verify the relation

$$\{\hat{Q}_-, \hat{Q}_+\}^* = 2\hat{H}. \tag{32}$$

where we have used the relations $[f, b]^* = [f^+, b^+]^* = 0$.

Then, defining the operators

$$\hat{Q}_1 = \hat{Q}_+ + \hat{Q}_-, \quad \hat{Q}_2 = -i(\hat{Q}_+ - \hat{Q}_-) \tag{33}$$

we find that

$$[\hat{Q}_i, \hat{H}]^* = 0. \tag{34}$$

and

$$\{\hat{Q}_i, \hat{Q}_k\}^* = 2\delta_{ik}\hat{H}, \quad i, k = 1, 2. \tag{35}$$

Note that the relations (34) and (35) are Lie-isotopic counterparts of the superalgebra relations (15) and (16) so that it can be refered to as a simplest realization of *iso-superalgebra*.

It is instructive to verufy that $\hat{Q}_i$ are iso-hermitian operators, $\hat{Q}_i^+ = \hat{Q}_i$. Let us take, for example, $\hat{Q}_2$:

$$\hat{Q}_2 = -iI^*(\hat{Q}_+ - \hat{Q}_-) = -i^T * (\hat{Q}_+ - \hat{Q}_-)$$

$$\hat{Q}_2^+ = -(\hat{Q}_+ - \hat{Q}_-)^+ * (i^T)^+ = -(\hat{Q}_+ - \hat{Q}_-) * (i^T) = \hat{Q}_2$$

where we have used the relation $A^+ = T^H A^H T^{-1}$ and $T^H = T$ (here, H denotes usual hermitian conjugation), and the fact that the iso-hermitian conjugation has the same properties as the hermitian one[3]; for example, $(A * B)^+ = B^+ * A^+$, $(A^+)^+ = A$, etc.

Therefore, the hamiltonian $\hat{H}$, being an iso-square of iso-hermitian operator, $\hat{H} = \hat{Q}_1 * \hat{Q}_1 = \hat{Q}_2 * \hat{Q}_2$, has indeed a non-negative energy spectrum, as it is displayed in (29).

It should be noted that the same element T must be used in the liftings of the commutation and anticommutation relations for bosonic and fermionic operators to provide a consistent lifting of the supersymmetry, $T = T_B = T_F$. This is in close analogy with the wellknown requirement that the frequences ω_B and ω_F must coincide to provide aupersymmetry of the hamiltonian H of the QM harmonic oscillator (see section 2). Perhaps, here a possibility arises that one can regain a supersymmetry by appropriate choice of T_B and T_F when the frequences ω_B and ω_F are posed not to be equal to each other; SUSY is explicitly broken, $\omega_B \neq \omega_F$, but iso-SUSY might be taken to be exact. However, in this case the lifting will be two-fold due to the fact that the commutation and anticommutation relations should be lifted by different fixed elements T_B and T_F, respectively. We shall not discuss this interesting possibility here noting however that the Lie-isotopic actually can provide regaining of the conventional symmetries, such as, for example, Lorentz one, recovering thus conceivable symmetry-breaking models; see for a review Ref. [7].

4 Supersymmetric hadronic mechanics

In this section we briefly discuss a possible Lie-isotopic way to generalize supersymmetric quantum mechanics.

Generalization of the hadronic mechanical SUSY oscillator which corresponds to a "free" HM theory, to the case including "boson-boson" and "boson-fermion" interactions can be made in an essentially similar way as it in the conventional free SUSY QM case. Namely, we note, first, that the HM SUSY generators $\hat{Q}_\pm$ will remain iso-nilpotent under the generalization

$$\hat{Q}_+ = B^-(b, b^+) *^+ f, \quad \hat{Q}_- = B^+(b, b^+) * f \tag{36}$$

where $B^{\pm}$ are arbitrary functions of b and b^+ (all multiplications are, clearly Lie-isotopic); $(B^-)^+ = B^+$ and $(B^+)^+ = B^=$. Hereafter, to simplify notation we drop the hat, indicating that the Lie-isotopic lifting has been made, over operators.

Under these definitions of SUSY generators, the associated hadronic mechanical supersymmetric hamiltonian, H, can be written as

$$H = \{Q_+, Q_-\}^*. \tag{37}$$

Indeed, since the operators $Q_\pm$ defined by (36) are iso-nilpotent, $Q_\pm * Q_\pm = 0$, we have

$$[Q_\pm, H]^* = 0. \tag{38}$$

Let us represent bosonic operators $B^{\pm}$ in the convenient form

$$B^{\pm} = B_1 \pm iB_2 \tag{39}$$

where B_i are assumed to be iso-hermitian operators. Further, in the matrix representation, fermionic operators f and f^+ can be presented as

$$f = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f^+ = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{40}$$

$$\sigma^{\pm} = \sigma_1 \pm \sigma_2$$

so that the wave function is two-component, with upper (lower) component corresponding to fermionic number $n_F = 1$ (0). Combining Eqs.(36), (39) and (40), for operators

$$Q_1 = Q_+ + Q_-, \quad Q_2 = -i(Q_+ - Q_-) \tag{41}$$

we have the following matrix representation:

$$Q_1 = B_1 * \sigma_1 + B_2 * \sigma_2, \quad Q_2 = B_1 * \sigma_1 - B_2 * \sigma_2, \tag{42}$$

It is easy to verify then that, again, as in the HM SUSY oscillator case (section 3), the following anticommutation relations hold:

$$\{Q_i, Q_k\}^* = 2\delta_{ik} H, \tag{43}$$

It is important to note that (43) does not depend on commutation relations between B_1 and B_2. Also, since the hamiltonian H defined by (43), (43) is not necessarily quadratic in b and b^+, it can describe an interaction.

After some algebra, for the hadronic mechanical supersymmetric hamiltonian (37), we have finally, in the matrix representation,

$$H = \frac{1}{2}\{B^-, B^+\}^* + [B^-, B^+]^* * \sigma_3 \tag{44}$$

where the σ-matrices have been assumed to obey the relations

$$[\sigma_i, \sigma_j]^* = 2i\hat{\epsilon}_{ijk}\sigma_k. \tag{45}$$

The hamiltonian (44) can be treated as a Lie-isotopic generalization of the hamiltonian of supersymmetric quantum mechanics[9, 10], and recovers it in the case $T \to 1$.

The first term in the SUSY HM hamiltonian (44) is pure bosonic and contains, therefore, only "boson-boson" interaction terms while the last one describes "boson-fermion" interactions.

We note from (44) that the hamiltonian is isosupersymmetric only then it contains fermionic degree of freedom, presented by the last term in (44), so that $[B^-, B^+]^* \neq 0$.

5 Discussion

One can try to specify further the matrix representation of the SUSY HM hamiltonian (44) by defining the bosonic operators $B^\pm$ in the form

$$B^\pm = \frac{1}{2}(\mp ip + W(q)) \tag{46}$$

in order to preserve a quadratic srtucture of the hamiltonian (44) in the mommentum p. If the momentum operator p and the superpotential W can be defined as iso-hermitian entities, the bosonic operators B^- and B^+ will be, evidently, conjugated to each other. Then, inserting (46) into (44) we get

$$H = \frac{1}{2}(p * p + W * W + i[p, W]^* * \sigma_3). \tag{47}$$

The last term in (47) can be presented also in the form $W' * \sigma_3$, where we have denoted $W' = \partial_q * W(q)$. This hamiltonian can be treated as a Lie-isotopic generalization of the hamiltonian of Witten's supersymmetric quantum mechanics[9].

The hamiltonian (47) can be treated also as a set of two conventional (not iso-supersymmetric) hamiltonians, H_+ and H_-,

$$H_\pm = \frac{1}{2}(p * p + W * W \pm W'(q)). \tag{48}$$

which have the same spectrum due to the underlying iso-supersymmetry, for arbitrary function $W(q)$.

It is then reasonable to expect that if the ground state is not degenerate, one can find, as in the conventional SUSY QM case[10] for Schrodinger equation, all energy levels, $E > 0$, for (one-dimensional) iso-Schrodinger equation with the hamiltonians of the type (48), by simple *iterative* procedure.

One of the open problems is to maintain a coordinate (superfield) representation of the Lie-isotopic anticommutation relations (25) of fermionic operators, f and f^+. In the conventional case, $f = \theta$ and $f^+ = \partial_\theta$ act on a supersymmetric wave function $\Psi(x,\theta) == a_1(x) + \theta a_2(x)$, where θ is a Grassmannian variable, $\{\theta, theta\} = 0$, and verify the relation $\{\theta, \partial_\theta\}\Psi(x,\theta) = \Psi(x,\theta)$. A naive way to generalize the Grassmann algebra of anticommuting variables might be to set the relation $\{\theta, theta\}^* = 0$, i.e. to state that θ's are *iso-nilpotent*, $\theta * \theta = \theta T \theta = 0$. However, T can not be here an element of Grassmann algebra for the obvious reason that the left-hand-side of the latter relation will be odd (anticommuting) while the right-hand-side is not. So, the generalization of Grassmann algebra seems to be provided obly by an even (not Grassmannian) entity T.

The problem of operating with Grassmann variables within the Lie-isotopic theory is of great importance, both on the grounds of its own meaning and also for future, practical purposes. For instance, Grassmann variables are known of essential use in the conventional, functional approach in quantizing of gauge field theories (standard $SU(2) \times U(1)$ electroweak theory, $SU(3)$ quantum chromodynamics) and, therefore, it seems to be reasonable to provide a Lie-isotopic version of Grassmann techniques to quantize Lie-isotopic gauge theory[11] (see for a review Ref [7]).

Acknowledgments

I would like to thank R.M.Santilli and D.F.Lopez for encouragement and very useful correspondence on Lie-isotopic theory, and L.V.Belousova for supprt during the preparation of this work.

References

[1] H.C.Myung and R.M.Santilli, Hadronic J. **5** (1982) 1277, 1367.

[2] R.M.Santilli, Hadronic J. **1** (1978) 574; **5**(1982) 1194 ;
H.C.Myung, Hadronic J. **1**(1978) 169.

[3] R.M.Santilli, Lett. Nuovo Cim. **38** (1983) 509.
R.Mignani, H.C.Myung and R.M.Santilli, Hadronic J. **6** (1983) 1873.

[4] *Proceedings of the Third Workshop on Lie-admissible Formulations,* Parts A-C (Hadronic Press, Nonantum, 1981);
R.M.Santilli, *Lie-Admissible Approach to Hadronic Structure,* Vol.II. *Covering of the Galilei and Einstein Relativities?* (Nonantum, MA., 1982);
Proceedings of the First International Conference on Non-Potential Interactions and Their Lie-Admissible Treatment, Parts A-D (Hadronic Press, Nonantum, 1982).

[5] R.Mignani, H.C.Myung and R.M.Santilli, Hadronic J. **6** (1983) 173.

[6] M.Nishioka, Lett. Nuovo Cim. **39** (1984) 369; **40** (1984) 77; **41** (1984) 377;
Lett. Nuovo Cim. **86 A** (1985) 151;
Hadronic J. **7** (1984) 1158, 1636; **10** (1987) 291, 195, 309; **11** (1988) 97.

[7] A.K.Aringazin, A.Jannussis, D.F.Lopez, M.Nishioka, and B.Veljanoski, *"Santilli's Lie-isotopic Generalization of Galilei's and Einstein's Relativities in Classical Mechanics: A Review of Their Mathematical Foundation, Physical Applications and Proposed Experiments".*
Preprint of the Inst. for Basic Research (Cambridge, Mass.) IBR-TP-901 (1990) 352 pp.

[8] A.K.Aringazin, *"Validity of the Pauli's Principle in the Exterior Branch of Hadronic Mechanics".*
Hadronic J. **13** (1990) 183-190.

[9] E.Witten, Nucl. Phys. **B 185** (1981) 513; **B 202** (1982) 253.

[10] F.Cooper and B.Freedman, Ann. Phys. (N.Y.) **151** (1983) 99;
E.Alvarez, Phys. Rev. **D 29** (1984) 320;
F.Karsch, E.Rabinovici, G.Shore and G.Veneziano, Nucl. Phys. **B 242** (1984) 503;
L.Gindenstein and I.Krive, Uspekhi Fiz. Nauk **146** (1985) 553. (in Russian)

[11] M.Gasperini, Hadronic J. **6** (1983) 935, 1462.